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A THEORY OF FATIGUE BASED ON
UNBONDING DURING REVERSED SLIP

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SUMMARY

Fatigue is interpreted as a progressive unbonding of atoms as a result of reversal of slip caused by cyclic loading. The s-N equation is derived by using an exponential equation for crack growth and assuming that failure will occur when the crack reaches an arbitrary depth. The effect of stress amplitude is introduced by using the term for inelastic strain from the Ramberg-Osgood empirical equation for the stress-strain curve. The resulting expressions agree with experimental data and afford a means of interpreting most of the known fatigue phenomena. They are also used to derive a new method of predicting the effects of loading of variable amplitude and to analyze the effects of a mean stress other than zero. The effects of notches are discussed in terms of stress-concentration factors. Other subjects treated include torsion, combined loadings, brittle fracture, ultimate tensile stress, temperature effects, surface effects, and corrosion fatigue. Recommendations for research are given in Part 7. Application of the theory to design and stress-analysis will be covered in a separate paper.

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PART I - DEVELOPMENT OF IDEAS

The failure of metals under repeated stresses of relatively low magnitude has long been a serious problem for engineers and has been the subject of intensive experimental and theoretical investigation. It is therefore unnecessary to give an extensive description of the phenomenon. Those not familiar with the subject will find an excellent summary of its various aspects in the recently published book "Fatigue and Fracture of Metals" (1952) edited by William Murray (Ref. 1).[*] Another valuable source of information is the book "Prevention of the Failure of Metals Under Repeated Stress" (1941) by the Battelle Memorial Institute (Ref. 2). Freudenthal (Ref. 3) has presented an analysis of fatigue (pp. 380 to 387) and many excellent suggestions on fatigue testing (pp. 560 to 573). One of the best references, from the engineering point of view is Timoshenko's "Strength of Materials" Vol. II, 2nd Edition (Ref. 4).

The author's own experiences with fatigue will be outlined briefly, because they played a part in the development of the theory which is proposed in this paper; they will also serve as a brief review of the problem as it affects aircraft design. The first contacts with fatigue were made while the author was employed by the Bureau of Air Commerce (now the Civil Aeronautics Administration) during the years 1930 to 1937. The failure of propeller blades and engine parts which had been in service for a relatively large number of hours became a serious problem. This led to more thorough inspection procedures and the removal of service nicks and scratches from propeller blades. (See pp. 2 and 3 of Ref. 2). Engine failures were reduced by more careful design and by the accumulation of experience.

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[*]References are listed on page 87.

While engaged in the revision of the airworthiness requirements
(about 1932), the author encountered the gust loading problem and began
to see possibility of fatigue of metal aircraft structures. The present
status of this problem has been excellently reviewed by Dryden, Rhode, and
Kuhn, in Paper No. 2, pp. 18 to 51, of Ref. 1. Fortunately, from the
fatigue standpoint, the aluminum alloys in use at that time were not very
"strong": they operated at relatively low stress levels and their duct-
ility tended to alleviate stress concentrations. Consequently no serious
fatigue problems arose in the primary structure.

During the years 1938 to 1947 the author was in charge of structural
analysis and research at the Lockheed Aircraft Corporation. By this time
higher-strength alloys were being used (24S-T4 and 14S-T6). At the present
time (1952) these have been replaced to a considerable extent by the alloy
75S-T6. Recognizing the increasing probability of fatigue failures, the
Lockheed Corporation authorized an extensive research program in which
fatigue data were determined and full-scale parts were tested under condi-
tions simulating actual service. One of the papers resulting from this work
is listed as Ref. 5. Other large aircraft companies also developed similar
programs and pooled their information through technical committees. As a
result, much valuable information became available to aircraft designers.

One of the significant things that happened during this period was the
application of failure-detection wires to fatigue specimens. This tech-
nique had originally been developed by the Lockheed Research laboratory
in order to determine the manner of failure of a large wing joint during

a static test. Mr Henry Foster applied the idea to fatigue specimens (Ref. 6) by cementing a fine wire across the specimen where it was expected to break. The wire was incorporated in a relay circuit which would stop the fatigue machine when the wire failed. It was found that the machines were stopping considerably before a crack could be observed in the specimen with the naked eye. Microscopic examination, however, usually disclosed a very fine crack at the point where the wire broke. (Mr. Foster's paper contains other valuable information and should be read by everyone interested in preventing fatigue failures.)

This idea has so far seemed to have received little practical application, but it could well be made the basis for detecting incipient failures of important structural parts in service. Perhaps the most important result of this work, so 'ar as its influence on the author is concerned, was the evidence that fatigue cracks actually formed long before they could be detected by ordinary means of observation.

The next clue came while the author was preparing a paper for the RAND Corporation (March 1949) which later became Chapter 16 of Ref. 7. The following is quoted from the book, p. 295:

"One more point is of interest in connection with the mechanism of plastic strain. In almost every textbook or paper dealing with plastic strain there appears a diagram, similar to Fig. 1, illustrating the physical action involved in slip. (Photographs of actual specimens confirm this. See Ref. 9*.) It can be seen that certain elements of the material, which were originally transmitting tension, are now unloaded. Therefore, they must have been broken away from their partners

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*Reference 31 of this paper

by a true tension failure. The actual local stresses developed during this action could be quite high; perhaps they would approach the true cohesive strength of the material."
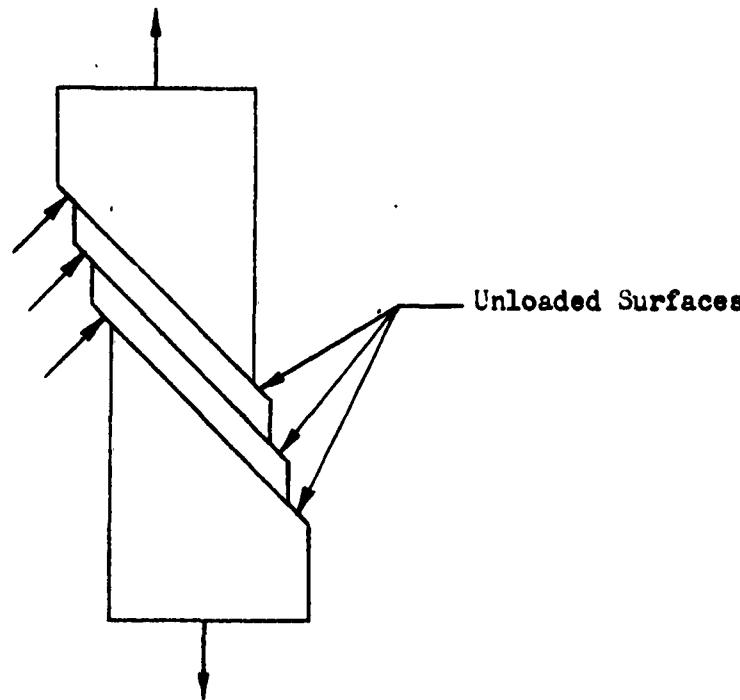
Fig. 1 - Separation Failures During Slip

On page 296 of Ref. 7 the following statement was made:

"The idea that cohesive failures may take place in the surface, at very low strains, seems to have some bearing on the subject of fatigue failure. It would seem that fatigue cracks are basically of the same type as the cleavage failures produced under conditions of low temperature, rapid loading, or low shear stress (or of combinations of these). Both phenomena appear to be forms of instability, the only difference being in the rate of propagation.* " (The second sentence is now believed to be wrong; see p. 65.)

The footnote which also appears on this page was added in 1949 while the original paper was being edited for inclusion in the book. It reads:

" *This idea has since been further investigated by the author. It can be imagined that a crack would be initiated by repeated loading in which a shear plane underwent a reversal of position upon each reversal of load. The basic assumption involved is simply that each atom, or element, becomes "inactive" after once being broken away from its previous partner during slip. On being forced back to its original position, it causes another element to become inactive. This action, repeated many times, would cause a crack that could eventually cause failure. Experimental investigation of this hypothesis is now (1952) under way."

The idea referred to here can be illustrated by a highly simplified model consisting of an orderly array of atoms as shown in Fig. 2.
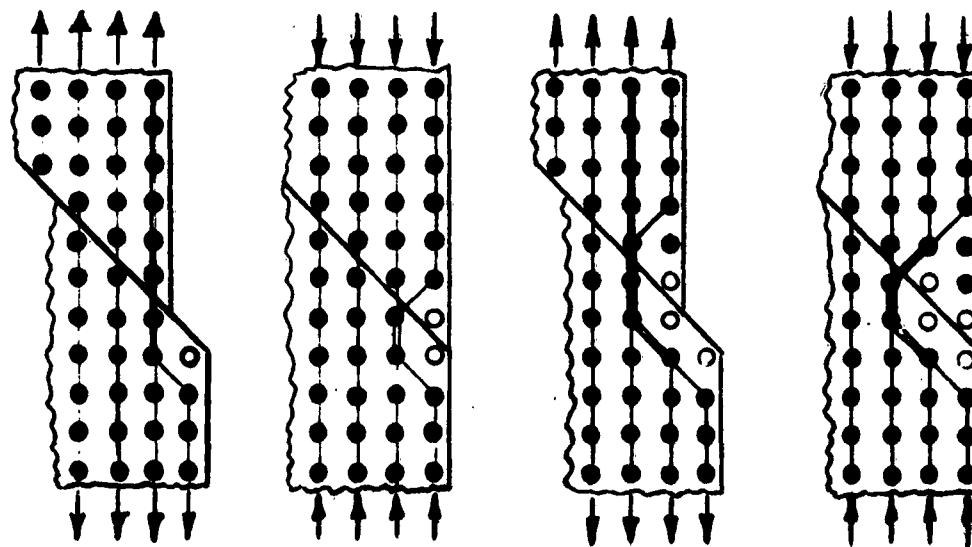

Fig. 2 - Formation of a Crack During Repeated Loading.

It may be imagined that when a tensile stress is applied slip occurs along
the plane of maximum shear stress (45 degrees). For simplicity assume
that only one atom slips out on the first application of load as shown
in (a). This atom (No. 1) becomes "unbonded," in the sense that it no
longer transmits a tensile stress directly to its previous partner. (This
is indicated by a hollow circle.) Now assume that the loading is re-
versed and that slip again occurs, causing the unbonded atom to return to
its previous position (Fig. 2 b). Having once been broken away from its
original partner, it is possible that it will not completely regain its bond:
the atom above it (No. 2) is therefore shown as unbonded. In Fig. 2c
the tensile force is again applied and slip is again assumed to move the
lower set of atoms to the right. Atom No. 2, having become unbonded,
will now cause No. 3, below it, to become unbonded also. In Fig. 2d the
process is carried one more step, causing a crack of two atoms width,
along the plane of slip. It is instructive to make a model of this type
by cutting a piece of paper along the 45° line and sliding it back and
forth. After a number of reversals the unbonded atoms, together with
those which drop out of the line of force transmission, will form a notch-
like region. The stress concentration effects can also be visualized in
terms of the number of bond lines which must be carried around the end
of the crack.

This elementary model (Fig. 2) demonstrates how a crack may form and
grow progressively with alternating loading, without any change in the
length of the specimen. The sketches also indicate roughly, by means
of lines drawn between atoms, how the atoms near the end of the crack
become overstressed as the crack deepens. This stress concentration
will obviously be a function of crack depth. It should also be noted that the
"free surface" essential to this theory, may occur within the specimen if there
is a discontinuity (flaw) in the material.

The extent of the slip which occurs on application of the load is a function of the shear stress acting across the slip plane. The shear stress is in turn a function of the tensile stress. (For the case shown it would equal one-half the tensile stress.) Hence the number of atoms over which the slip occurs during each reversal of load will depend on the magnitude of the applied stress. It is not necessary to assume that all of these atoms become unbonded in the manner previously described, but it would seem logical to assume that a certain fraction of the atoms which are broken away do not regain their bond when they are returned to their previous positions. It might also be expected that the unbonding tendency would be affected by the value of the mean stress, chemical effects from the surrounding media, , etc. These ideas will be discussed later.

The action just described appears to be the key to an explanation of fatigue failure. Such action could result in fatigue of a perfect crystal, but most engineering materials are actually composed of many small crystals having random orientation of slip planes. For such materials it can be assumed that a fairly large fraction of the surface crystals will be oriented in such a way that the plane of maximum shear stress coincides with the weakest slip plane of the crystal. The other crystals, being differently oriented, will not exhibit the same degree of slip; some of them may be located so that their action, under the stresses applied, will be virtually elastic. It is therefore highly probable that the crystals in which the most slip occurs will be surrounded by an aggregate of crystals whose overall behavior may be described as partially elastic. This situation is now generally accepted and is the basis for the Orowan theory

of fatigue (Ref. 8), which will be discussed in Part 6. The crack-forming mechanism shown in Fig. 2 must therefore be considered as acting in conjunction with residual stresses, for all polycrystalline materials. This is not a necessary part of the theory, for completely reversed loading, but it appears to explain why fatigue cracks can be formed under a **cyclic** loading in which the stress cycle remains entirely in tension. This will be further discussed in Part 4.

The theory had been developed to this point in the spring of 1952, when the RAND book (Ref. 7) was being prepared for publication. The next step which was planned involved a series of fatigue tests, (to be made at the University of California) in which replicas of the specimen surface would be made at various intervals during the test, with the specimen under **static** tension. After a crack had been **located**, the replicas would be examined under the electron microscope. It was thought that by this method the crack might be traced back to the early stages of loading, thus tending to dispel the idea, commonly held, that the crack does not begin to form until after a certain number of loading cycles have been applied.

This investigation has not been completed, but a recent (1952) preprint of an A.S.T.M. paper by Craig (Ref. 9) gives the results of some electron microscope studies of the development of fatigue failures. The following quotation (from p. 9) is of interest:

"While the resolving power of even the electron microscope is not sufficient to determine the exact course of crack initiation, there is good evidence in the materials examined that the initial cracks seem to grow from local deformed or fragmented paths that follow the general direction of slip bands on the surface."

This does not prove that the cracks start from the very beginning, but it does show that the cracks are associated with slip. (See also Ref. 10, pp. 53 and 54). It should be noted that the type of crack which would be developed by the mechanism illustrated in Fig. 2 would probably not be visible in the early stages, even under the electron microscope, because failure to bond would not in itself cause the crack to remain open after the load had been removed. In fact, any residual compressive stress would tend to force the surfaces of the "crack" together, even under zero external loading. One could expect to see the crack only after fragmentation or excessive local deformation had occurred, which is exactly what the photographs of the Craig paper indicate. It would appear also that fragmentation would tend to result from unbonding and would have to be preceeded by a "crack" which would be invisible.

The final step in the development of a theory for fatigue was precipitated by the publication of a paper (T.N. 2787) by the National Advisory Committee for Aeronautics (Ref. 11). The authors, Hardrath and Utley, reported the results of an experimental investigation designed to test Miner's theory (Ref. 12) for repeated stresses of varying amplitude. They found that the theory gave good results for a sinusoidal type of loading, but appeared to overestimate the fatigue life for a loading cycle of an exponential nature. These results indicate that higher stresses should have been given greater weight in the integration process.

Because of the importance of this problem in aircraft design, the author decided to attempt a mathematical formulation of the crack-growth theory previously described, to be used as a basis for developing an improved theory for repeated loading of varying amplitude. By defining

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[*] See Appendix 1 of Supplement to P-350.

fatigue as the development of a crack of constant depth and by using the well-known Ramberg-Osgood empirical expression for inelastic strain an equation for the s-N diagram was obtained, for cyclic stresses of constant amplitude. This had the typical form that has long been accepted as an empirical method for representing test results. A new method of predicting behavior under stresses of varying amplitudes was then obtained. This showed an improvement over Miner's method. The derivation of these equations will now be outlined, after which the theory will be applied to other aspects of the fatigue problem.

PART 2 - DERIVATION OF FATIGUE EQUATIONS
FOR REVERSED LOADING OF
CONSTANT AMPLITUDE

In Part 1 it was shown how a crack could be caused by failure of some of the atoms to bond after they had once been moved to the surface of the material by slip caused by shear stresses. It was also indicated that the rate of crack growth would tend to increase with increasing crack depth. The latter tendency suggested a relationship of exponential form, such as

$$h = A e^{an} \qquad (1)$$

where   h = crack depth

A = a constant

α = a factor depending on the stress amplitude

n = number of cycles of reversed loading.

To indicate the physical nature of the problem, a hypothetical curve of crack depth __versus__ cycles of repeated loading is shown in Fig. 3.
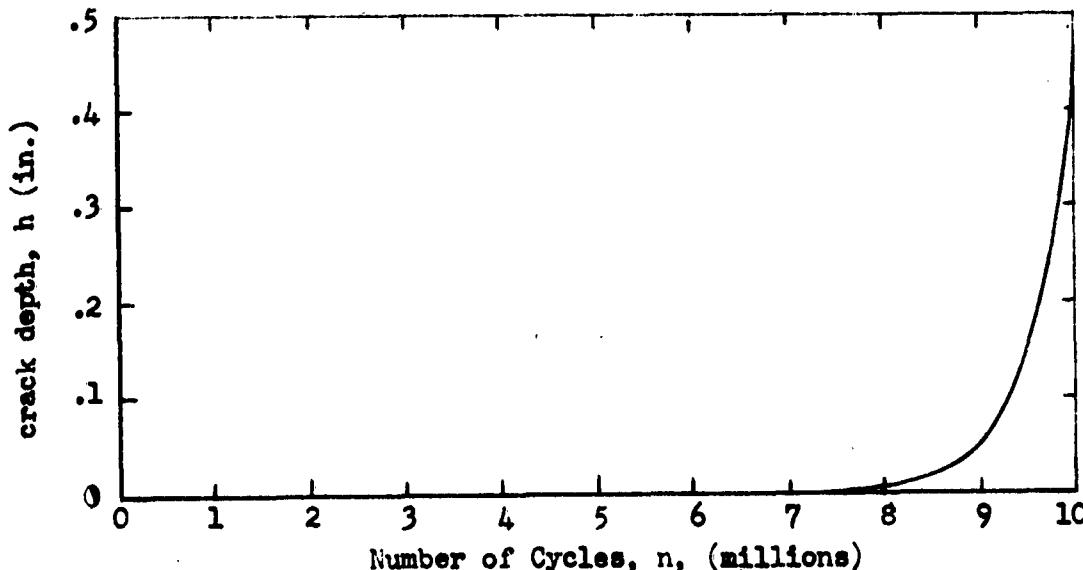

Fig. 3 Diagrammatic Representation
of Crack Growth

If it is assumed that the crack (or unbonding) starts from the beginning, the rate of growth <u>must</u> be very slow during a large portion of the loading history. For example, assume that the part breaks when the crack depth reaches one-half inch and that this occurs after 10-million reversals. The distance by which the crack deepens on each reversal of loading is obviously of extremely small magnitude.

It is known, however, that the rate of crack growth just prior to failure is relatively rapid. (See Appendix 27, Ref. 2; also Ref. 13.) Wilson and Burke (Ref. 14) have made careful measurements which show that a crack growth of 0.6 inch will be produced by about 30,000 cycles of reversed loading, once the crack has been started.

An important lesson may be learned by attempting to plot these results on the last portion of the growth curve of Fig. 3. It is quite obvious that the line would appear to be vertical. The fact that the growth curves obtained in Ref. 14 appeared to be linear is of course consistent with the fact that they represent only a very small segment of the entire curve. This shows that the function involved must be of a type which will exhibit a very slow growth for a long period, followed by a relatively rapid rate of growth. An exponential growth law satisfies this condition.
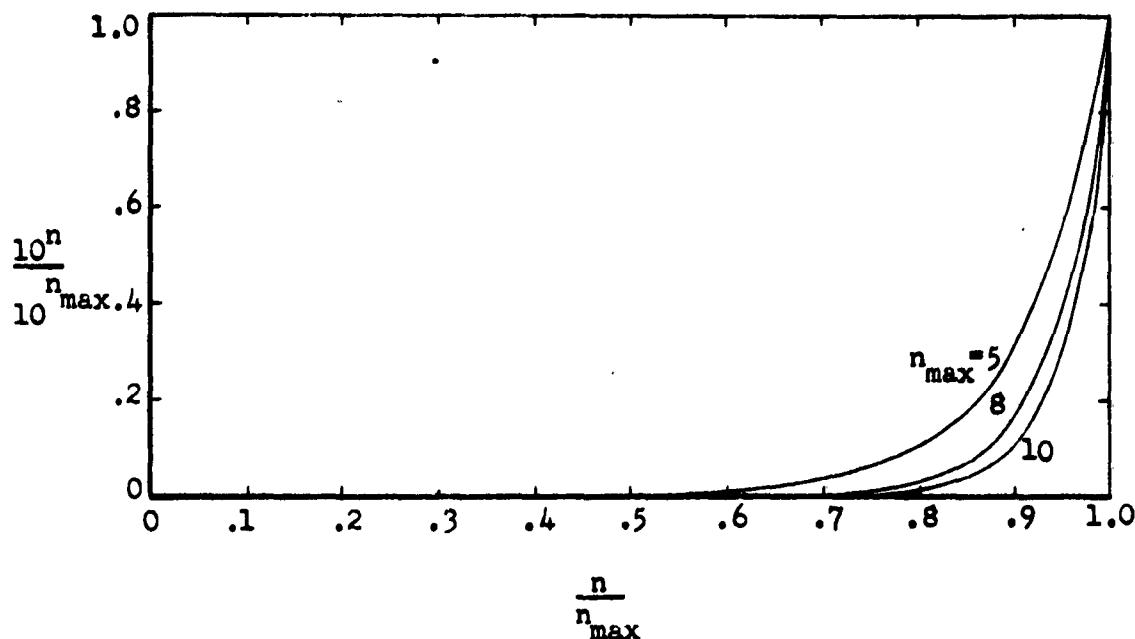
Fig. 4 Dimensionless Plot of 10^n

To illustrate this graphically the values of 10^n have been plotted non-dimensionally in Fig. 4, using three different maximum values for n. It can be seen that the abruptness of the curve depends on the maximum value of the exponent used.[*]

Another thing revealed by this reasoning is that the particular type of function used is not of great importance, so far as determining the final depth of the crack is concerned. Any function which could be fitted to experimental crack growth data would give about the same results, after integration.

The steepness of the crack growth curve just prior to failure led to the idea that a fatigue failure criterion might be taken to be the attainment of a crack of arbitrary depth. Since the crack depth curve must be

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[*]The term n in Fig. 4 corresponds to the term an in Eq. 1.

practically vertical for a considerable number of cycles preceding failure, the particular value chosen to represent failure will have virtually no effect on the accuracy of prediction of the fatigue life, at least at low stress levels.

The physical significance of the term A, in Eq. (1) is found by setting n equal to zero, which gives h = A. The term A therefore represents the initial depth of crack,[*] while the exponential part of the equation may be thought of as a multiplier which shows how A grows during repetition of the stress cycle. In the first derivation of the fatigue equation A will be considered to be a constant. This assumption will later be modified in order to obtain more accurate expressions for _rate_ of growth.

The term α represents the initial rate of growth and must therefore be governed by the number of atoms that become unbonded during each stress cycle, at the very early stage of crack growth. This was assumed to depend on the value of the maximum shear stress, which for axial loading or bending is equal to one-half the axial stress. For such loadings, therefore, α may be considered to be a function of the maximum tensile and compressive stresses (assuming complete stress reversal). For combined loadings the maximum shear stress, or some related stress, would be used directly.

The amount of slip which occurs during a simple tension test may be accurately represented by the empirical term for inelastic strain in the Ramberg-Osgood equation (Ref. 15):

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[*]This interpretation may appear to be inconsistent, since n would be unity for a single cycle. However, it may be reasoned that n actually represents the number of reversals which occur _after_ the first loading has been applied. This would amount to replacing n by n-1 in all equations for crack depth.

$$\epsilon = \frac{s}{E} + C s^x \tag{2}$$

from which $\quad \epsilon_p = C s^x \tag{2a}$

where $\quad C$ = a constant

$\quad s$ = the tensile stress = P/A

$\quad x$ = an exponent determined by curve-fitting[1]

$\quad \epsilon_p$ = plastic (inelastic) strain.

It appears logical to assume that the same form of relationship would govern the degree of slip in the microscopic regions involved in fatigue, since the inelastic strain measured in a tension test is actually the integrated result of the individual slips which occur throughout the specimen. Equation (2) will therefore be substituted for the term a, with the following understanding:

1. The stress s actually represents the average or effective value of the shear stress which controls the degree of inelastic strain reversal.

2. Stress concentration effects may cause the slip in the critical region to exceed the inelastic strain experienced by the bulk of the material (this is further investigated in Part 5).

3. The number of atoms permanently unbonded during a reversal will not necessarily be the total number unbonded in the slip process. The ratio of permanently unbonded atoms to the total involved in slip will be assumed to be constant for the present.

For a specimen having no large stress concentration effects all of these items may be taken care of by adjusting the value of C, which may

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[1]. In Ref. 15 the term n is used instead of x. The change was made here to avoid confusion with the use of n for number of cycles.

therefore have a different value from that obtained in tension tests.

Substitution of the value of α from Eq. (2a) gives the following expression for Eq. (1), with h_o representing some arbitrary constant value of crack depth:

$$h_o = A e^{C s^x N} . \tag{3}$$

(Note: n has been changed to N, which denotes fatigue life.)

Taking the natural logarithm of both sides gives

$$\ln h_o = \ln A + C s^x N \tag{a}$$

from which
$$s^x N = \frac{\ln h_o - \ln A}{C} . \tag{b}$$

The right-hand term may be replaced by a constant, B, giving

$$s^x N = B . \tag{c}$$

From this the equation for the s-N diagram can be written in either of two familiar forms

$$\boxed{N = \frac{B}{s^x}} \tag{4a}$$

$$\boxed{s = \frac{B'}{N^{1/x}}} \tag{4b}$$

where
$$B' = B^{1/x}$$

By taking the common logarithm of both sides of Eq. (4b) one obtains

$$\log s = \log B' - \frac{1}{x} \log N \tag{4c}$$

This gives a straight-line equation when N and s are plotted on log-log graph paper.
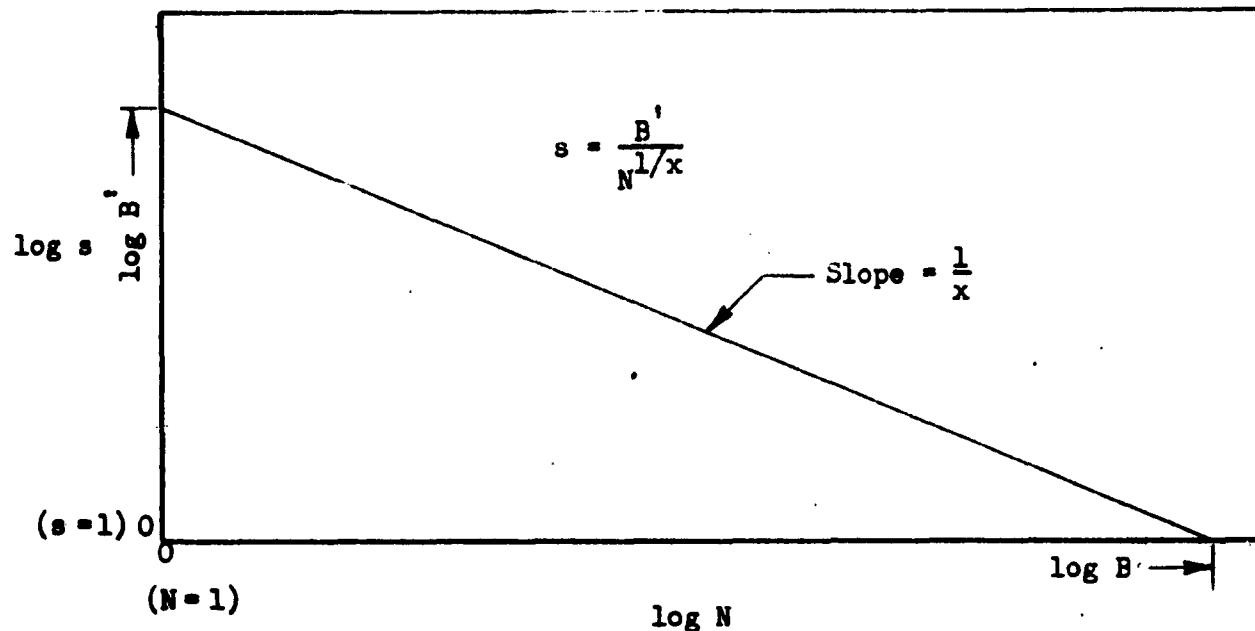
Fig. 5 Logarithmic Plot of s - N Equation

It has long been known that a log-log plot of test data for completely reversed repeated loading can be fitted by straight lines, well within the scatter of the data. For example, Peterson (Paper No. 4 of Ref. 1) shows a collection of fatigue data plotted on a log-log basis (from Weisman and Kaplan). The mean values, as well as the scatter band for the test results, are shown to correspond to the straight-line relationships.

To illustrate more clearly the significance of Eq. (4), the data from NACA TN 2798 (Ref. 11) are plotted in Fig. 6, as presented in the original paper. This diagram is a semi-log plot, the log scale being used only for N. This type of plotting has become customary, probably because it is convenient to condense the N scale, but inconvenient to read stress from a

Fig. 6 s-N Curves for Tests of 24S-T4 Rotating-Beam Specimens (From NACA TN-2798)
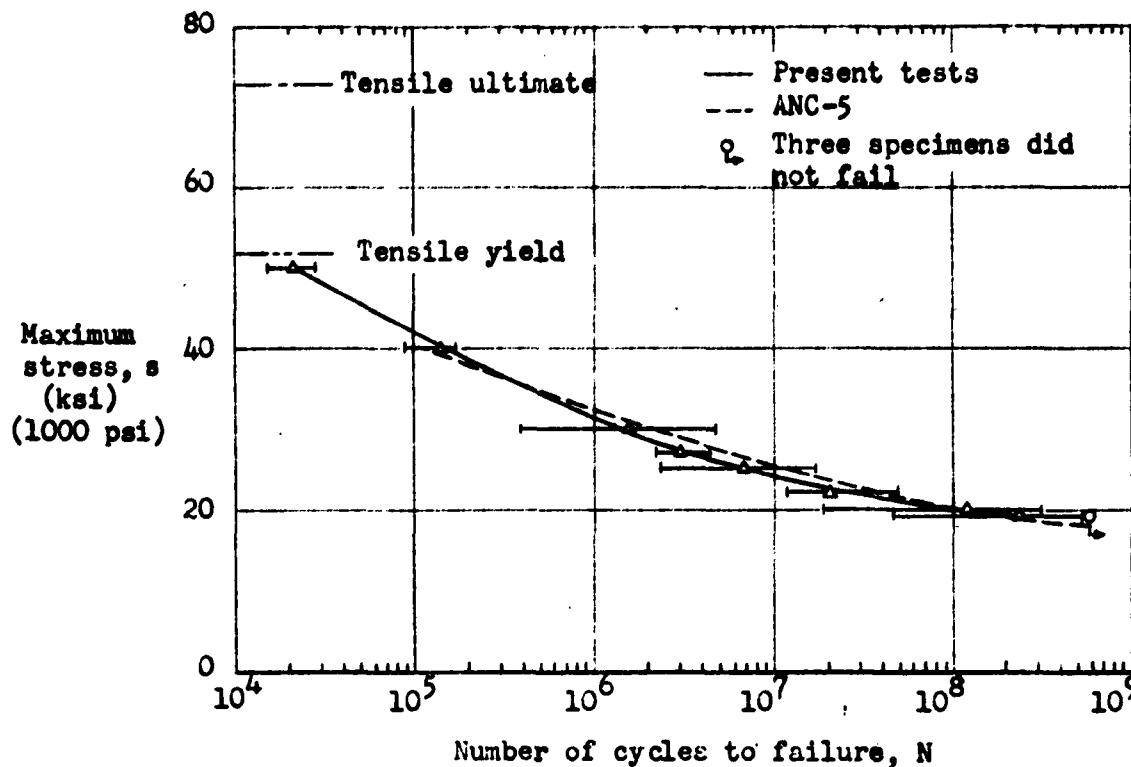
logarithmic scale. The mean values (indicated by triangles on Fig. 6) are replotted on a log log scale in Fig. 7. In order to emphasize that
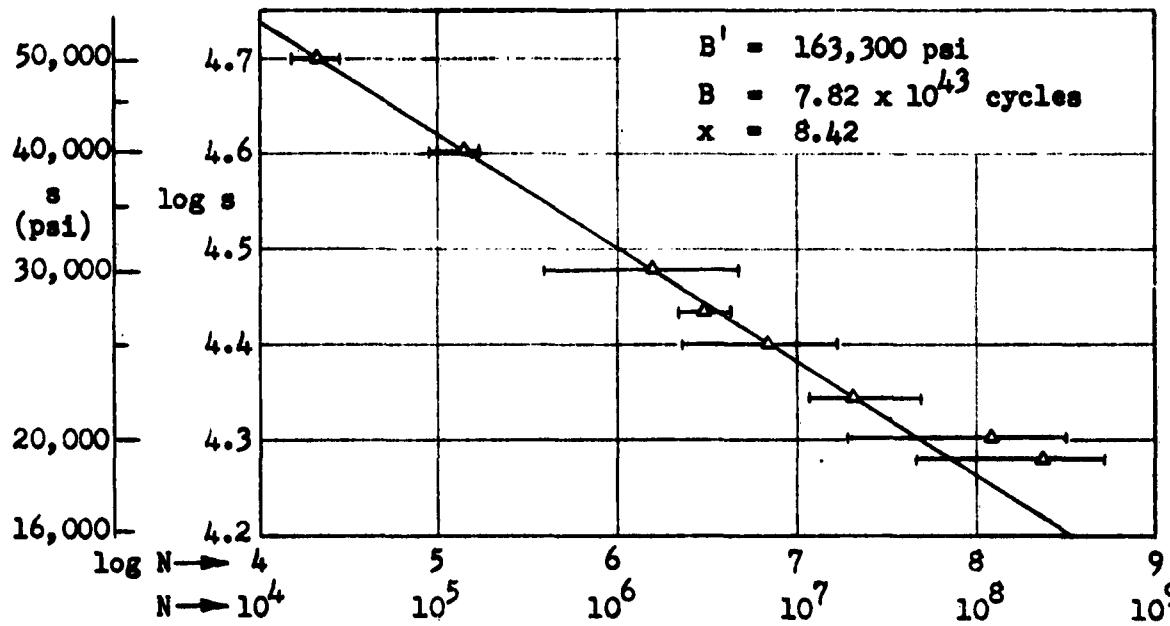

$B' = 163,300$ psi

$B = 7.82 \times 10^{43}$ cycles

$x = 8.42$

Fig. 7 Log-Log Plot of Fig. 6

the actual values are not being plotted, but rather their logarithms, the scales are shown in terms of logs. The actual values of stress are indicated by an auxiliary scale, which is non-linear.

It is evident that on the log-log plot a straight line gives an almost perfect fit for the points representing the averages of from 10 to 20 specimens at each level (101 specimens in all). The slope(-1/x) gives a value of $x = 8.42$. A tendency for the curve to flatten out at high values of N may be observed. This indicates a tendency toward an "endurance limit" (to be discussed later). It will be apparent, however, that any attempt at accurate curve-fitting in this region is virtually meaningless, in view of the large scatter of the data (over one million cycles variation in about seven million). This scatter must be expected because of the nature of the phenomenon and it has an important bearing on design for fatigue.

While the log-log plot is useful in checking theories such as represented by Eq. (4), it does not convey a proper sense of proportion to the observer. It is unfortunate that fatigue data are so seldom plotted on ordinary (Cartesian) graph paper. The use of a semi-log plot is particularly unfortunate, because it serves no useful purpose other than condensing the N-scale; one might as well lookinto a distortion mirror so far as obtaining a true picture is concerned!

Figure 8 shows the data of Figs. 5 and 6 plotted on a linear basis, using a scale covering 100,000,000 cycles. Even this range does not include the last two points on Fig. 6. A strikingly different picture is obtained. The allowable stress for a given lifetime drops very rapidly as the required number of cycles is increased from one to about one million.

$$N = \frac{B}{s^x}$$

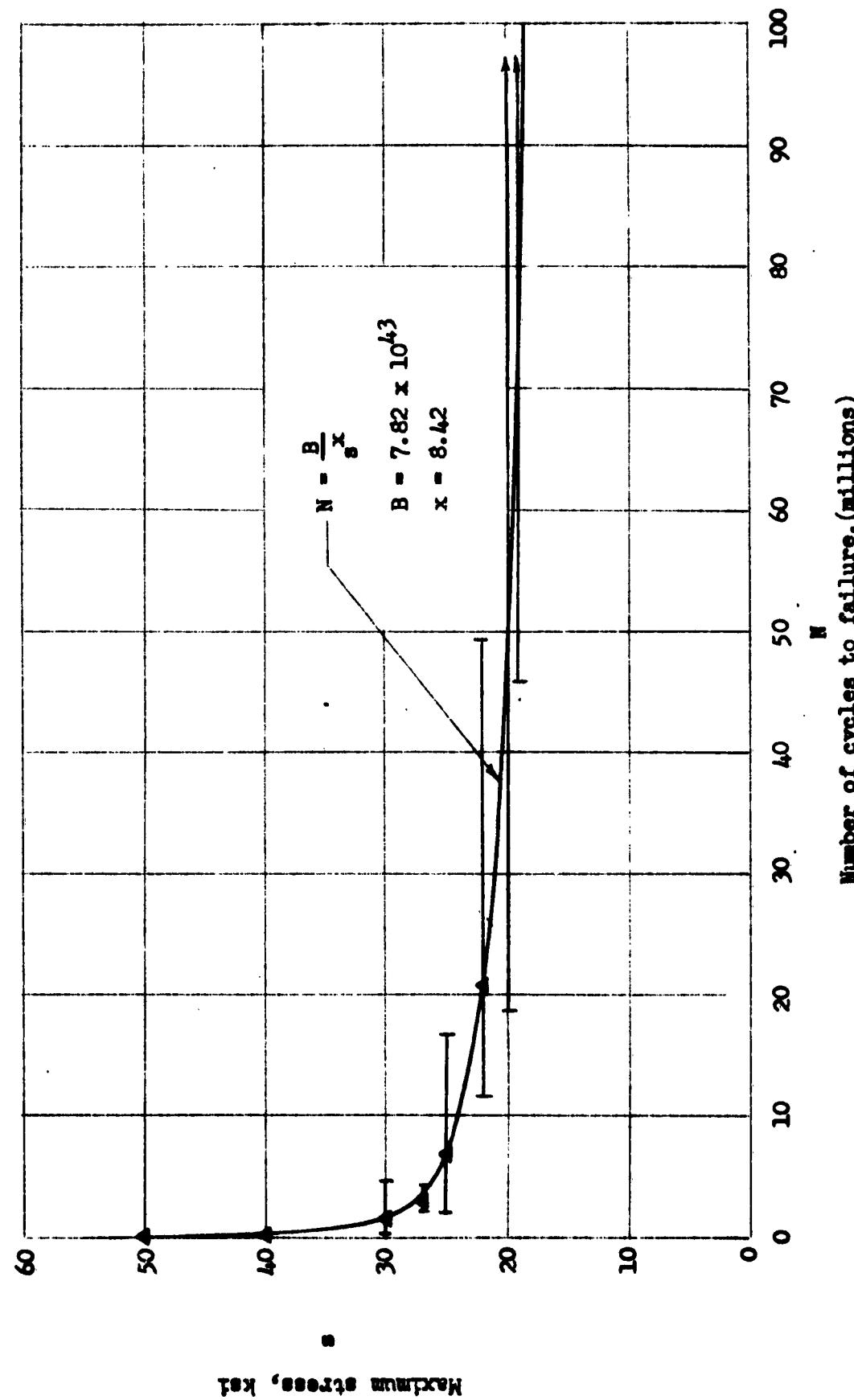$$B = 7.82 \times 10^{43}$$

$$x = 8.42$$

Fig. 8 - Linear Plot of Fig. 6.

From one million to ten million the drop is more gradual, while at values of N beyond ten million the stress decreases at a very low rate.

A curve obtained from Eq. (4) has been plotted on this figure. Values of B and x.were determined so as to produce a good fit in the early part of the curve $(B = 7.8 \times 10^{43}; \ x = 8.42)$. The ability of Eq. (4) to fit the test data is obviously equal to, if not better than, that obtained from the accepted formulas for other types of failure (modulus of rupture in bending, column failure, etc.).

The extreme scatter of the test data at large values of N is now seen to constitute a relatively narrow scatter band, so far as the values of stress are concerned. On the other hand, the scatter in terms of N is so great that any attempt to predict the lifetime at low values of stress appears to be virtually meaningless. (The importance of this observation in connection with design for fatigue is discussed in the Supplement.)

It is interesting to note that the exponent x, as determined from the fatigue curve by means of Eq. (4), is about 8.4. The value determined by the Ramberg-Osgood formula, from tension and compression tests of the material, was approximately 25.* This would indicate that the exponent tends to be lower in reversed loading than in undirectional loading. (There is experimental evidence to substantiate this; see Fig. 294, p. 460, Ref. 4.) It should also be noted that the fatigue tests were conducted in bending, not axial loading. If it could be shown that the inelastic strain exponent for reversed loading has a definite relationship to that obtained in the routine tension test, it would provide a means of predicting fatigue behavior from a tension test alone. (The true stress at ultimate load would be used to determine the value of the constant, B, as explained later.)

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*Based on information supplied by the NACA.

In applying the basic equation for crack growth to the case of variable stress amplitude (Part 3), it was found that the assumption of a constant value for A might need to be improved. Since A represents the magnitude of the crack depth produced on the first loading cycle, it would be logical to assume that it is also a function of the applied stress, s. If A is now replaced by the term $A' s^x$, the equation will become

$$h_o = A' s^x e^{C s^x N} \tag{5}$$

where A' is a new constant, possibly having the same value as C.

After taking the natural logarithm of both sides the equation is:

$$\ln h_o = \ln A' + x \ln s + C s^x N \tag{a}$$

from which

$$s^x N = \frac{\ln h_o - \ln A' - x \ln s}{C} \tag{b}$$

or

$$N = \frac{B_2 - C_2 x \ln s}{s^x} \ . \tag{6}$$

This equation is similar to Eq. (4a) except that the constant B is replaced by a term which represents a decreasing value of B as s increases. This means that in Fig. 5 the curves for different values of s would not all have the same intercept on the log N axis. In view of the fact that the experimental data can be fitted quite satisfactorily by Eq. (4), there is no need to use Eq. (6) for this purpose, especially since it requires the determination of one more constant. The reason for including this refinement will be explained in Part 3.

It is well known that for some materials there appears to be a stress level below which fatigue failures cannot be obtained. This value of the maximum applied stress is called the endurance limit. On a linear chart (such as Fig. 7) this would be indicated by a horizontal line at the endurance limit and the s - N curve would be asymptotic to this line. On a log-log chart (Fig. 7) the transition from the straight-line portion of the curve to the horizontal portion would appear to be more abrupt. This impression is of course the result of the distorted scale used.

There has been much speculation concerning the significance of the endurance limit, but the most important fact is that there is such a limit for some materials, at least so far as can be determined from tests. When this is true, it is possible to design a part so that, for all practical purposes, it will never fail in fatigue.

For many materials, such as aluminum alloys, there appears to be no well-defined endurance limit. These same alloys also appear to have no well-defined proportional limit.* On the other hand, materials such as low-carbon steel, which do have an endurance limit, also exhibit a well-defined proportional limit. This is one of the facts that has led many investigators to suspect that fatigue is caused by the inelastic ("plastic") strain.

There are several ways in which an endurance limit can be interpreted, in the light of the theory just presented. These are illustrated in Fig. 9.

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*Recent information from Mr. R.L. Templin, of the Aluminum Research Laboratories, indicates that high-strength aluminum alloys do have a proportional limit which is as well-defined as that of low carbon steel.
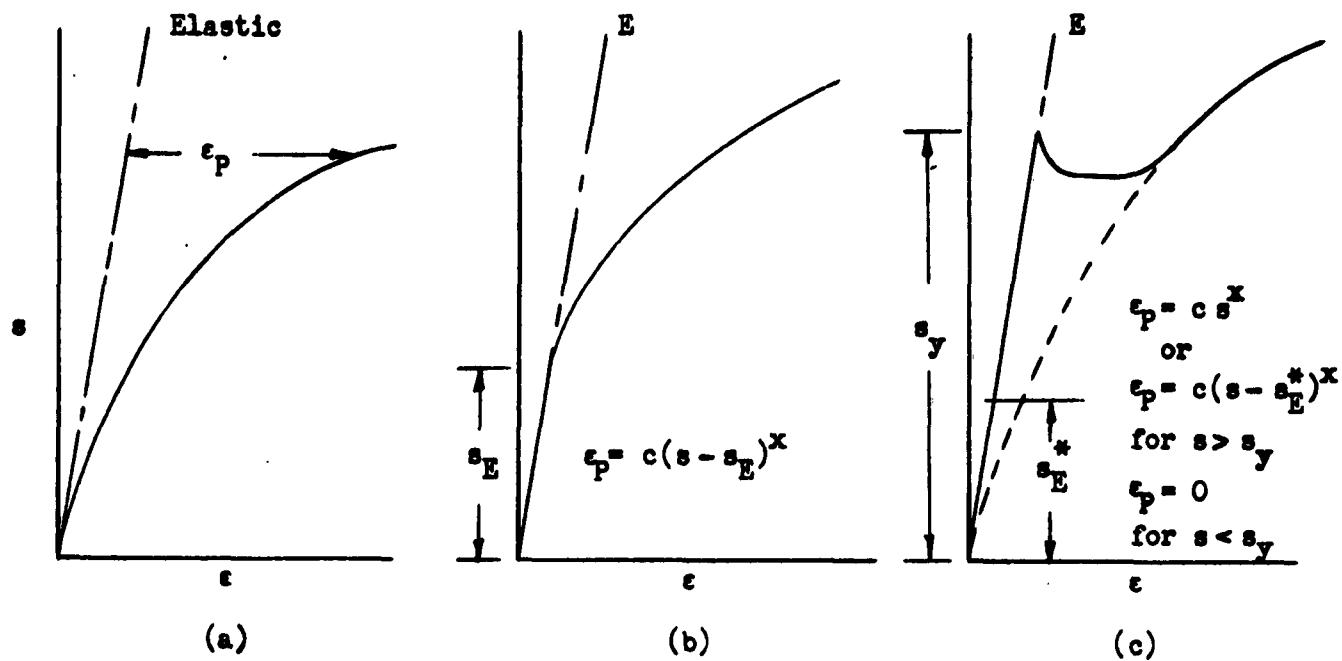
Fig. 9 Different Ways of Interpreting
the Endurance Limit

Fig. 9(a) shows the type of inelastic behavior assumed in deriving

the $s-N$ equations, up to this point. Inelastic strain (shown to a greatly

enlarged scale) is assumed to occur at any stress greater than zero. In

Fig. 9(b) an elastic limit (s_E) is assumed, below which no inelastic strain

occurs. At stresses greater than s_E inelastic strain is assumed to occur

according to the empirical relationship originally assumed (Eq. 2). It

can be seen that the Ramberg-Osgood equation (Ref. 15), which corresponds

to Fig. 9(a), could be modified to correspond to Fig. 9(b). Since the

strains given by the equation for low values of stress are very small, the

modified equation could no doubt be fitted to the experimental stress-

strain curves without difficulty. The values for C and x would of course

have to be altered. It should be noted also that the original Ramberg-Osgood

equation, while fitting stress-strain diagram quite well for strains which can be measured, does not prove the non-existence of an elastic limit. Hence the type of stress-strain diagram shown in Fig. 9(b) must be considered as a possibility, even for relatively "soft" materials. Another interpretation might be that inelastic strain at very low values of stress is the result of unavoidable flaws or weak crystals, having a statistical distribution. The existence of such strains would then appear to be of no practical significance. The assumption of an elastic limit would then correspond to the common engineering practice of ignoring the branches of the Gauss frequency distribution curve, beyond certain limits.

Figure 9(c) represents the well-known type of stress-strain diagram in which no observable plastic slip occurs up to some particular value of stress, at which a very rapid slip of considerable amount takes place. This is generally considered to mean that the metal contains bonds which prevent slip entirely up to a certain stress and then suddenly break. The shape of such curves suggests that the basic curve may be either of type (a) or type (b); that is, without the special bonding elements the curve might follow the dotted line of Fig. 9(c).

One other mechanism must be considered: "strain-hardening". (The author quotes this term because he feels it is often misused, as explained on page 267 of Ref. 4.) Gensamer has given an excellent description of this phenomenon (pp. 1 to 3 of Ref. 1), which may be summed up by quoting a portion of his description:

> "By strain hardening we mean the increase in the elastic
> range, produced by strain. This is not the elevation of the
> elastic limit observed in monotonous increase in strain, most

of which kind of strain hardening is reversible, as was so nicely shown by Dalby and others many years ago. As we put the metal through successive cycles, the elastic range gradually increases and the area within the loop gradually diminishes. Figure 1, from the work of Gough, illustrates this effect. The loop area measures the work of plastic deformation. If the specimen does not break first, it strain-hardens to the point where it undergoes purely elastic action; further cycling does nothing."

Figure 1, on page 2 of Ref. 1, shows the results of experiments in which the strain was completely reversed many times. The loop appears to have become a straight line at about 400,000 cycles, after which it remains straight.

For materials having a sharp yield point (Fig. 9(c)) there is evidence that the elastic portion of the curve breaks down in the early stages of high cyclic loading (see Ref. 4, p. 460). This results in a stress-strain loop which would correspond to the dotted line portion of the stress-strain diagram in Fig. 9(c).

Regardless of the exact reason for the apparent increase in the elastic range, it can be seen that it will have an important influence on the endurance limit. As the stress cycle is repeated, the inelastic strain decreases, provided that the stress remains within certain limits.

During the time when plastic slip is still occurring, cracks would probably be formed as previously described. These cracks would affect only a very small amount of material, since they remain small during most

of the lifetime. The bulk of the material would at the same time be undergoing a decrease in the amount of inelastic strain reversal. Since the magnitude of the slip experienced by any crystal is largely controlled by the deformation of the entire specimen, such slip would stop as soon as the specimen became completely "elastic", if that is possible.

This explanation offers one more method by which an apparent elastic range may be established. It is sufficient to state that, according to the theory here proposed, <u>fatigue cannot occur when the action is purely elastic</u>. An alternative and perhaps better way to state this is: <u>fatigue cannot occur in the absence of slip</u>. Although there seems to have been relatively little experimental investigation of fatigue of brittle (elastic) materials, the absence of any evidence to the contrary seems to be in accordance with the above statement. (See also the discussion in Part 6.)

Since it is evident that there are various ways in which an elastic range may be established under cyclic loading, the effect of such a range on the shape of the s-N diagram (Eq. 4) will now be investigated.

For a material with a well-defined yield point, as shown in Fig. 9(c), the effect will be simply to cut off the s-N diagram at some value of stress, giving a sharp break on the log-log plot (Fig. 7). The break would be imperceptible on the linear plot (Fig. 8). The value at which this occurs is known to be considerably less than the yield point, which can probably be attributed to stress-concentration effects at the surface.

For a material having an elastic limit, followed by gradual slip, as shown in Fig. 9(b), Eq. (4) will also apply if the term s is replaced by $(s - s_E)$. The elastic limit used here will actually represent the endurance limit, which includes the effect of stress concentrations, etc.:

$$N = \frac{B}{(s-s_E)^x} \qquad (7a)$$

$$s = s_E + \frac{B'}{N^{1/x}} \qquad (7b)$$

Since new constants C and x must be used in the modified Ramberg-Osgood equation, the value of B will also be different from that of Eq. (4). Equation (7) will plot as a straight line on log-log paper if $(s-s_E)$ is plotted instead of s. If s is plotted, the curve will "sag" and will become asymptotic to s_E.

To illustrate the shape of the curve obtained from Eq. (7), this equation has been plotted on a log-log basis in Fig. 10. A good fit to the average test values is again obtained, including the points at large values of N. It therefore appears that the material is behaving as shown in Fig. 9(b) and that the apparent endurance limit is about 14,000 psi.
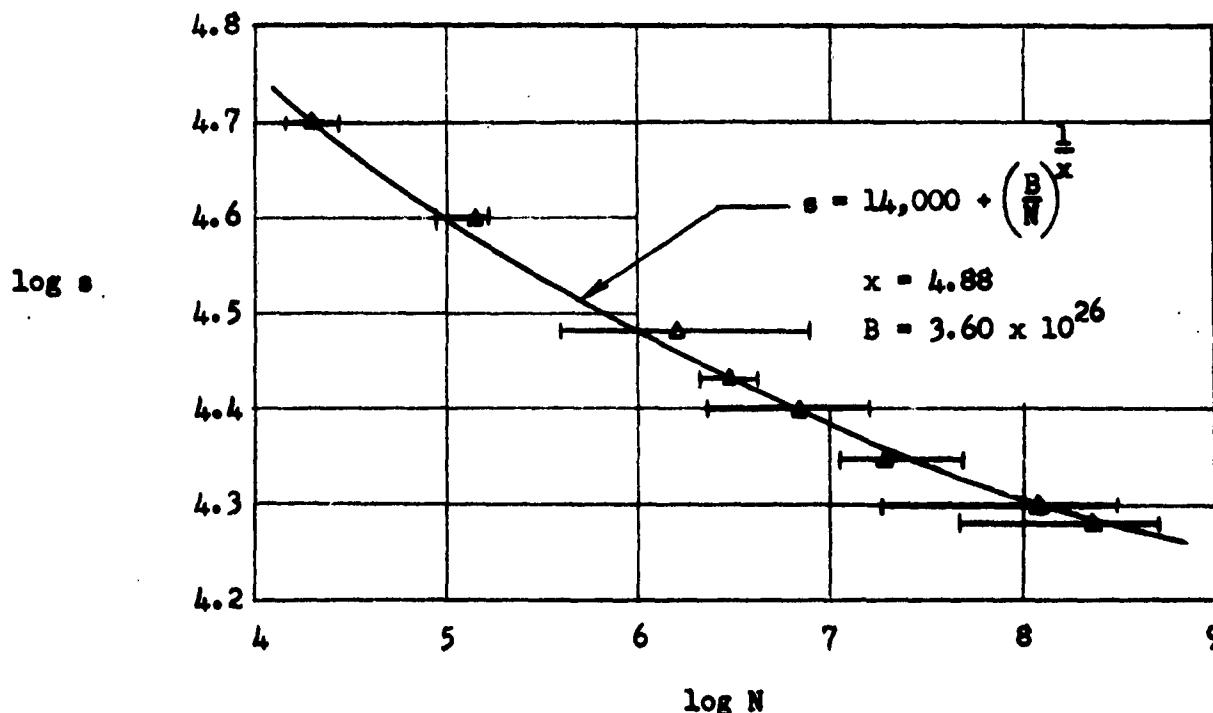

Fig. 10 - Effect of An Elastic Limit
(Test Data from NACA TN-2798)

PART 3. EFFECT OF STRESSES OF VARYING AMPLITUDE

There are many situations in which the designer is faced with the problem of repeated loading of varying amplitude. Two general types of variation may be considered. The one of most interest to airplane designers is that in which a certain "spectrum" of stress applications may be assumed to be repeated over and over again. The number of cycles involved in the spectrum will be very small, relative to the total number to failure.

The second type of loading consists of a large number of repeated loadings of constant amplitude, followed by another loading of different amplitude, this second loading being continued until failure occurs. This may be expanded to include three or more different loadings, but the essential difference is that in the first case the spectrum covers only a small part of the fatigue life, while in the second case a single spectrum covers the entire life.

The situation which exists in the "short-spectrum" loading is shown in Fig. 11a for completely reversed (balanced) loading.
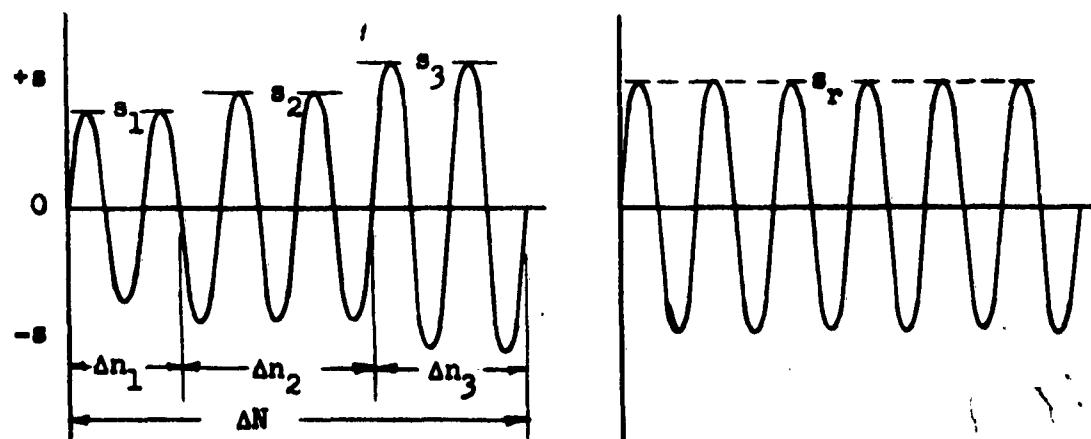
(a) Short-Spectrum　　　　　　　　(b) Equivalent Reduced Stress

Fig. 11 Variable Stress Amplitudes Represented by a Spectrum

In actual service, the spectrum would have a random distribution of stress amplitudes, but by statistical analysis this may be represented by an orderly arrangement such as shown in (a). For a very short spectrum the order in which the stresses are applied will (according to the theory to be developed) have no practical significance. There is some speculation, however, on the effects of a random distribution of stress amplitudes (see Ref. 1, p. 223). (The method of analysis which follows could probably be extended to such cases by estimating the amount of unbonding which would occur when a stress cycle of any given amplitude is followed by one of a different amplitude.) The problem is to find the value of the stress which, applied at constant amplitude, will produce the same fatigue life as that which would be obtained with the stresses of variable amplitude. Figure 11b shows this diagrammatically. The stress s_r will be called the reduced stress.

One of the first papers on this subject was that of Miner(?)

(Ref. 16). He considered the process of failure to be divided into two parts, the formation of the crack and the growth of the crack. By assuming only that the function relating crack growth to stress is the same at any stress level he developed the basic principle involved in attacking this problem. Briefly stated, this consists in determining the fraction of the fatigue life utilized at any given stress level and adding these fractions. Failure is assumed to occur when the sum of these fractions is equal to unity. This is actually a special application of the interaction curve method now widely used to predict failure under combined loading (see pp. 341 to 345 of Ref. 17).

Miner's method (Ref. 12) assumes in effect a straight-line inter-
action curve (actually an i-dimensional surface) which can be expressed
by the equation:

$$\sum \frac{n_i}{N_i} = 1 \qquad\qquad (8)$$

where

n_i = cycles at stress amplitude s_i

N_i = fatigue life at constant stress
amplitude s_i

The member is expected to fail when the summation equals unity. If
the sum is less, it indicates that some service life remains.

In Ref. 1, Paper No. 10, N.M. Newmark has presented a treatment of
this subject in which, without actually using any particular theory of
fatigue, he has covered almost all aspects of the problem in a general way.
Perhaps the most important feature of his paper is the introduction of non-
linear relationships based on "damage" and "healing" curves. These terms,
which have been widely used in fatigue literature, are of course entirely
general and serve primarily as definitions of the unknown, not as explanations.

Most of the methods proposed for solving this problem result in a
prediction regarding the fraction of fatigue life that has been utilized.
For reasons which will be explained later, the author feels that this
approach is not suitable for design or stress analysis purposes. It would
be more satisfactory to be able to determine the stress amplitude at which
a certain fatigue life would be assured. For loading of variable amplitude
this can best be done by determining a reduced stress, as already described

and as illustrated in Fig. 11.

In Ref. 18, Clinedinst (1937) derived an expression for the reduced stress based on the general fatigue equation (Eq. 4 of this paper), which had already been determined experimentally. The expression for the reduced stress will now be derived by using the crack-growth equations from Part 2. This will show how further refinements may be made which may more nearly represent the conditions that exist and which will give better agreement with test results.

The method used will be essentially the same as that outlined by Newmark on p. 210 of his paper (Ref. 1) except that the objective will be to determine a reduced stress instead of a "cumulative cycle ratio". Consider two curves representing the growth of a crack at two different stress levels, as shown in Fig. 12 (not to scale).
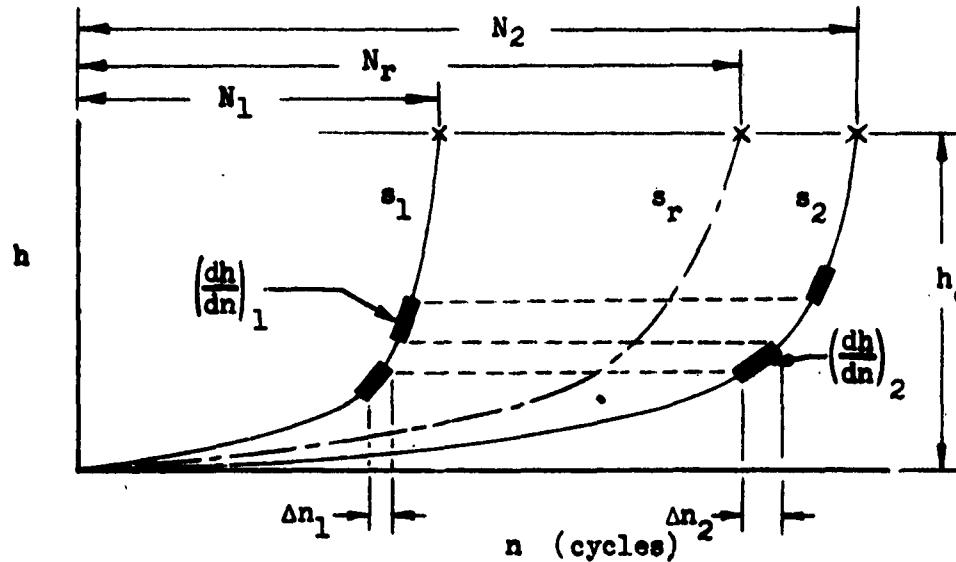

Fig. 12 Crack Growth for a Stress Spectrum Consisting
of Two Different Amplitudes

It will be assumed that stress s_1 is applied for Δn_1 cycles, after which stress Δ_2 is applied for Δn_2 cycles. The number of cycles in the spectrum $(\Delta n_1 + \Delta n_2)$ is assumed to be very small in comparison with the fatigue life N. The spectrum is to be repeated over and over again until "failure" occurs, at crack depth h_o. The actual crack growth curve for the combined loadings, as yet unknown, is illustrated by the dot-dash line labeled s_r. The problem is to find the value s_r which, if applied at constant amplitude, will cause failure after the same number of cycles, N_r. (Note that N_r is not equal to $N_1 + N_2$).

Since it was assumed (in Part 2) that the rate of crack growth was a function of crack depth, the effective rate of growth, at any value of n, will be determined by the weighted average of the rates for s_1 and s_2, at the same crack depth. This would be expressed mathematically by

$$\left(\frac{dh}{dn}\right)_r = \frac{\Delta n_1\left(\frac{dh}{dn}\right)_1 + \Delta n_2\left(\frac{dh}{dn}\right)_2}{\Delta n_1 + \Delta n_2} \tag{9}$$

For the case in which several different stresses are applied during the spectrum (Fig. 11) the above equation may be expressed as

$$\left(\frac{dh}{dn}\right)_r = \frac{\sum \Delta n_i\left(\frac{dh}{dn}\right)_i}{\sum \Delta n_i} \tag{10}$$

The rate of crack growth at any given stress level will be found by differentiating the expression for the depth of the crack with respect to n. Two cases will be examined. Assume first that the crack depth is

given by Eq. (3):

$$h = A \, e^{C \, s_i^x \, n} \tag{11}$$

The rate of crack growth is

$$\left(\frac{dh}{dn}\right)_i = A \, C \, s^x \, e^{C \, s^x \, n} \tag{12}$$

The effective rate, from Eq. (10), would therefore be

$$\left(\frac{dh}{dn}\right)_r = A \, C \left(\frac{\sum \Delta n_i \, s_i^x}{\sum \Delta n_i}\right) e^{C \, s_r^x \, n} \tag{13}$$

In this equation s_r has been used in the exponent, because by definition s_r would produce the same depth of crack as the stress spectrum.

The corresponding expression, in terms of s_r, is

$$\left(\frac{dh}{dn}\right)_r = A \, C \, s_r^x \, e^{C \, s_r^x \, n} \tag{14}$$

From Eqs. (13) and (14) it is evident that

$$s_r^x = \frac{\sum \Delta n_i \, s_i^x}{\sum \Delta n_i} \tag{15}$$

from which

$$s_r = \left[\frac{\sum \Delta n_i \, s_i^x}{\sum \Delta n_i}\right]^{1/x} \tag{16}$$

This is the desired expression for the reduced stress. It means that all stresses in the spectrum are weighted to the x-power and in proportion to their relative frequency of occurrence.

It is convenient to express Eq. (16) in terms of dimensionless factors. This may be done by selecting a base stress in the spectrum, s_o, and defining the stress ratios and frequency ratios by

$$r_i = \frac{s_i}{s_o} \tag{17}$$

$$k_i = \frac{n_i}{\sum n_i} \tag{18}$$

The reduced stress may now be expressed in terms of the base stress:

$$s_r = \beta s_o \tag{19}$$

$$\beta = \left[\sum k_i r_i^x\right]^{1/x} \tag{20}$$

(In computing β it is convenient to select as base stress, s_o, the lowest value in the stress spectrum, thus avoiding negative logarithms.)

The value of x will be determined by the basic s - N diagram in the region over which the stresses vary.

If there is an endurance limit,[*] s_E, it could be assumed that any stress below this value will not cause growth of the crack. This would be taken care of by substituting zero for s_i in Eq. (16), or for r_i in Eq. (20). The possible helpful effects of "understressing" will not be

[*](for constant amplitude stress reversals)

accounted for by this procedure, but there appears to be no proof that such understressing (repeated loading below the endurance limit) actually improves fatigue resistance when applied as part of a loading spectrum. Hence it is advisable to ignore such effects in computing the reduced stress for the short-spectrum type of loading, until there is experimental proof that such effects exist. In fact, there seems to be no proof that the endurance limit exists under such conditions. If this limit depends on not exceeding the elastic limit at any time, or on building up a quasi-elastic state, it is quite possible that an occasional high stress cycle will "wash out" all the helpful effects caused by the lower stress cycles. It would seem that the only safe procedure in computing the reduced stress would be to ignore the endurance limit, until further experimental evidence is obtained.

Having determined the reduced stress for a given loading spectrum, the fatigue life may be found by using the regular s-N diagram for constant stress amplitude.

Eq. (20) was applied to the sample cases on pages 18 and 19 of NACA TN-2798 (Ref. 11). No significant differences in the results were obtained, as compared with Miner's method. This is to be expected, since Eq. (16) actually represents Miner's method in the form of a reduced stress. From the NACA tests it appeared that the higher stresses required greater weighting. This led the author to reexamine the expression used for the depth of crack (Eq. (3)). A different expression was obtained by assuming that the initial depth (on first application of load) was a function of the inelastic strain

(Eq. (5)). The derivative of Eq. (5) is

$$\left(\frac{dh}{dn}\right)_1 = A' \, C \, s^{2x} \, e^{C \, s^x \, n} \qquad (21)$$

If this equation is used in the development of an expression for the reduced stress, Eqs. (16) and (20) become

$$s_r = \left[\frac{\sum \Delta n_1 \, s_1^{2x}}{\sum \Delta n_1}\right]^{\frac{1}{2x}} \qquad (22)$$

$$\beta = \frac{s_r}{s_o} = \left[\sum k_1 \, r_1^{2x}\right]^{\frac{1}{2x}} \qquad (23)$$

The new expressions obviously give greater weight to the higher stresses and therefore tend to bring the theory into closer agreement with test data. (See Supplement for results of using this theory.)

Further empirical modifications could be made, if desired, to attach even greater weight to the higher stresses. No attempt will be made here to explore the secondary effects which might result from repeated stresses of variable magnitude. For the short stress-spectrum it would seem that any distribution of stress amplitudes[*] could be adequately represented by the method outlined above. In making tests, it is important that the number of cycles in the spectrum be held to a relatively low value.

Up to this point the variable stress amplitudes have been assumed to occur within a short stress-spectrum which is repeated until failure occurs. A different situation is found in the second type of loading, where the

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[*]Including random distribution

loading history is composed of only a few stages, each occurring at a
constant stress level. In the simplest case, consider s_1 to be applied
for n_1 cycles, after which s_2 is applied until failure occurs. The result
to be expected is shown diagrammatically by Fig. 13 (not to scale).
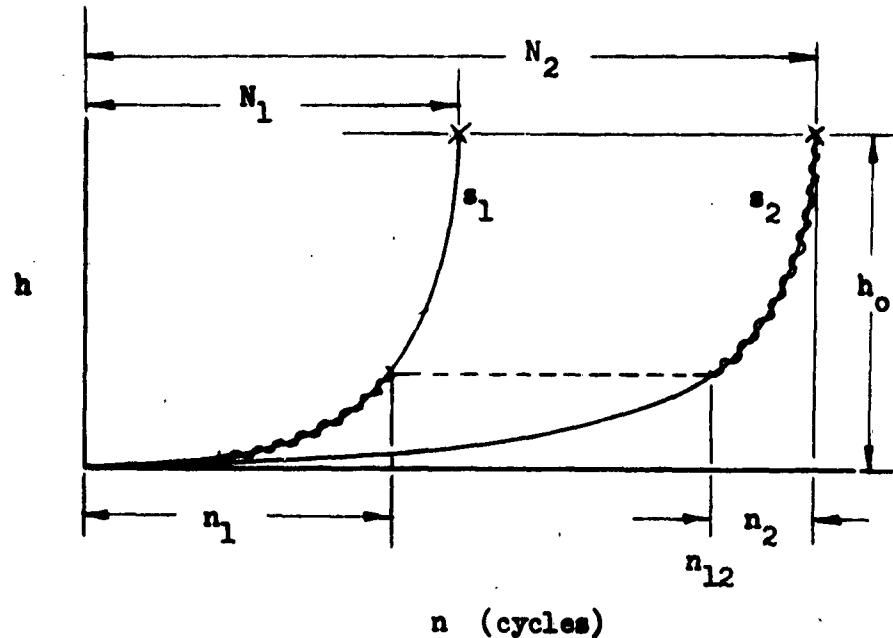
Fig. 13 "Overstressing", in Terms of Crack Growth

During n_1 cycles at stress s_1 the crack grows along the s_1 curve.
When the stress is changed to s_2 the transfer to the s_2 curve must be
made at constant crack depth. It is therefore possible to find a value
of cycles n_{12} which would have produced the same depth of crack at stress
level s_2. If Eq. (3) is used as a basis for crack growth, the resulting
relationship will be

$$n_{12} = \left(\frac{s_1}{s_2}\right)^x n_1 \qquad (24)$$

where

$$n_{12} = \text{equivalent life at stress } s_2, \text{ for } n_1 \text{ cycles at } s_1$$

If this relationship is used to find the total life, it will give the same result as the method previously developed for short-cycle loading, based on Eq. (3). This is because the same shape of curve was used for both stress levels (see Newmark's discussion of this, p. 207, Ref. 1). This means, for example, that if one-half the life at s_1 is used up, one-half of the life at s_2 will remain. This would be true, however, only when affine crack growth curves are assumed.

If Eq. (5) is used as a basis, the result will be

$$n_{12} = \left(\frac{s_1}{s_2}\right)^x n_1 + \frac{x \, \ell n \left(\frac{s_1}{s_2}\right)}{s_2^{\,x} \, C} \tag{25}$$

This expression is the same as Eq. (24), except for the added term. When s_1 is greater than s_2, this means that a larger fraction of the fatigue life at s_2 will be used up by "overstressing" at s_1. When s_1 is less than s_2 a negative logarithm is obtained, showing that "understressing" would not use up as much life as predicted by Miner's or any similar theory.

The situations described above are not very likely to be met with in aircraft design. Furthermore, there are undoubtedly other effects which occur when a long period at one stress level is followed by a change to a different stress level. The attention which has been devoted to these phenomena appears to be based on the hope that they would throw further

light on the nature of fatigue itself. Since Eq. 25 appears to explain the observed behavior, at least qualitatively, no attempt will be made in this paper to develop the equations any further.

It is known that a period of cyclic loading at a stress level just below the endurance limit may raise the fatigue life for subsequently applied stresses of greater amplitude. (See Refs. 1 or 2.) This would seem to be explainable by the fact that repeated loading at a low stress amplitude tends to cause the bulk of the material to approach an elastic state, as previously described. If this cycling is carried out below the endurance limit, the rate of crack growth will gradually decrease and a stable condition will be attained by the time the elastic state is reached. Subsequent loading at a higher stress level must therefore be based on a higher elastic limit, as shown by Fig. 9 and Eq. (7).

This explanation serves to emphasize the previously-expressed warning that under random cyclic loading the helpful ("healing") effect of "understressing" should not be included in the analysis.

PART 4 — EFFECTS OF UNBALANCED LOADING
(MEAN STRESS NOT EQUAL TO ZERO).

In Part 2 the type of loading assumed was that in which the loading alternates between equal values of tension and compression, with a mean stress of zero. For convenience, this will be referred to as balanced loading. When the mean stress is not zero, the term unbalanced loading will be used. In an airplane, for example, the wing is subjected to an average loading which is very nearly equal to the weight of the airplane. During flight through rough air the load fluctuates above and below this mean value as the result of up and down gusts (see Paper No. 2, Ref. 1). It is therefore important to extend the theory of fatigue to include the unbalanced type of loading. The same situation applies in a bridge structure, except that the mean stress is likely to be considerably higher, relative to the fluctuating stress.

The maximum and minimum values reached during a stress cycle will be assumed to remain constant, as in Part 2. The effects of varying amplitude have been analyzed in Part 3, for balanced loading. Similar methods can be developed for unbalanced loading.

Fig. 14 shows a case in which the stress varies between zero and some maximum value in tension, which will be designated as s_{max}. Assume that at some region in the structure there is a large-scale stress-raiser having an effective value of K. The actual stress will therefore vary as shown by the dashed curve in Fig. 14 (b). (Note that K also raises the mean stress.)
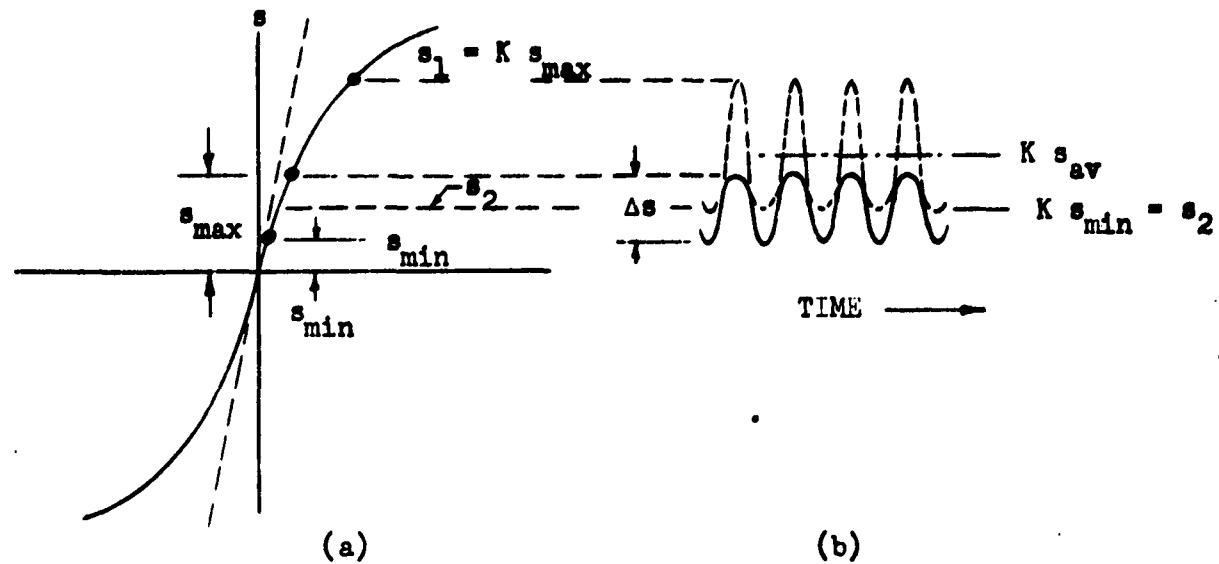
Fig. 14 Unbalanced Loading

It will be assumed, for the moment, that large scale stress-con-
centration effects have been accounted for and that we are dealing with
the actual stress in a region surrounding the crack (notch effects etc.
will be further discussed later). The stress-strain picture implied by
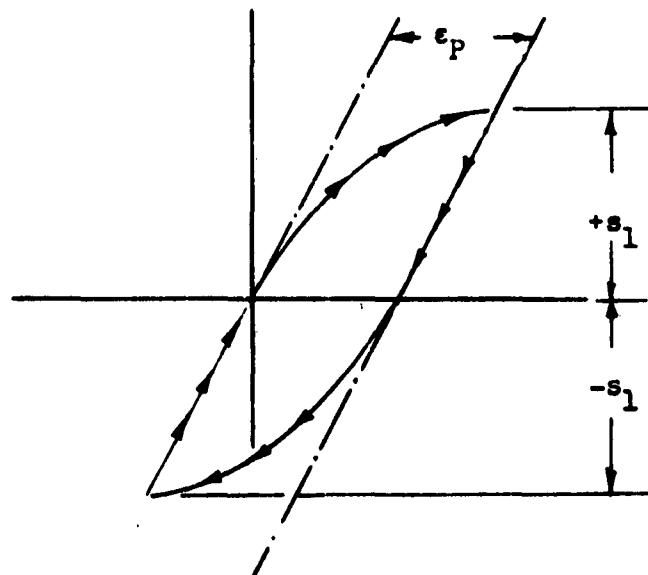the assumptions used for balanced loading (Part 2) is shown in Fig. 15

Fig. 15 Stress-Strain Loop for Balanced Cyclic Loading.

This type of action is known to occur and has already been referred
to (Ref. 1, pp. 1 to 3, see also Ref. 4, p. 460 and Ref. 10, p. 27).
Actually, however, the amount of slip which is involved, during crack
growth, will be much less than the amount determined by applying to Fig. 15
the stress-strain curves obtained in a simple tension test. As discussed
in Part 2, the loop may even narrow to such an extent that it becomes a
single straight line, representing elastic action. (This idea was used to
explain how ah inelastic material might develop an endurance limit.)

If this same reasoning is now applied to the unbalanced type of load-
ing a question immediately arises. It is generally considered that un-
loading is a purely elastic action (this applies only for time-independent
stress-strain relationships however; see the author's discussion in Ref. 7,
p.283). If the loading cycle remains entirely in the tension range, there
would appear to be no stress-strain loop at all and there would then be no
crack growth. This is illustrated in Fig. 16, for stresses varying between
s_1 and s_2.
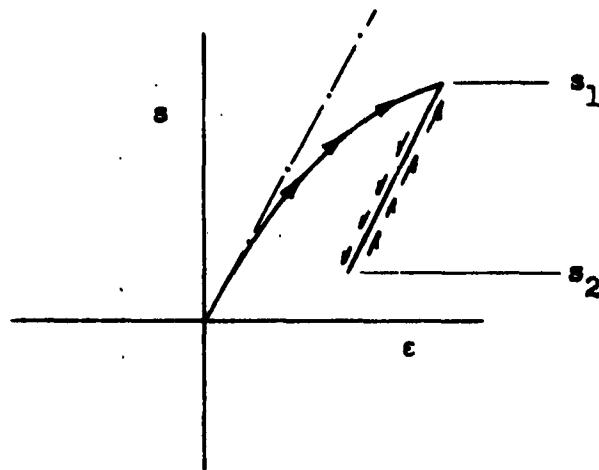
Fig. 16 Unbalanced Cyclic Loading

It is known, however, that fatigue can occur under such conditions. There are at least two explanations for this, both of which no doubt apply to some extent. First, it is known that the "law" of purely elastic action on unloading is not strictly true, for very small strains such as those involved in fatigue. Hence there may still be a loop. Second, it is possible for a "weak" crystal to undergo reversal of strain even though it might be imbedded in a medium that is otherwise purely elastic. This is illustrated by the extremely simplified picture in Fig. 17. Assume
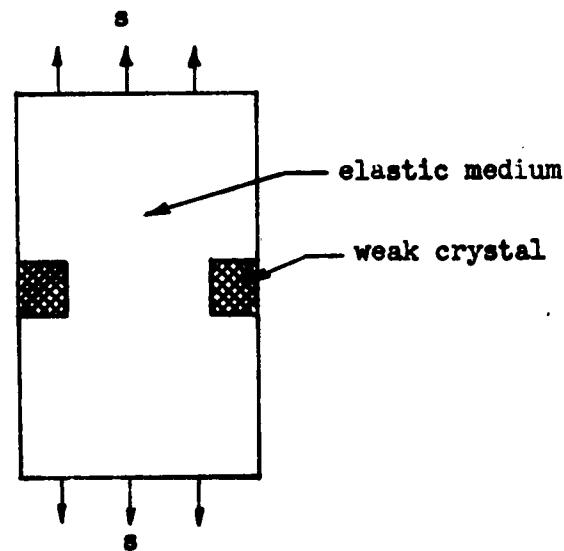
Fig. 17 Slip of a Crystal in an Elasti Medium

that the weak crystal starts to slip at a relatively low stress, or that it has an inelastic stress-strain relationship such as described by Eq. (2). When the bulk stress (over the entire cross-section) is varied between s_1 and s_2, the crystal must undergo the same change of length as the bulk of the material. If it slips at all, it will have to slip back again, even though the bulk stress remains tension. The small forces required to cause this slip will have no appreciable effect on the bulk stress.

In an aggregate composed of both "weak" (plastic) and "strong" (elastic) crystals the situation will be somewhat similar, except that the overall behavior of the material will not be totally elastic in nature. Instead it will depend on the integrated behavior of very large numbers of crystals having a random distribution of slip planes. This is the basis for the Batdorf-Budiansky theory of inelastic behavior (Ref. 19), which the author believes to be the most realistic theory yet proposed.

To determine the behavior of an actual material under repeated loading, it will be necessary to apply the principles of this theory to the case of cyclic loading. This would require knowledge of the behavior of single crystals under repeated loading involving a large number of cycles and would involve determining the residual stresses which result from non-uniform behavior of the individual crystals. Such a theory, if it were available, would no doubt explain why a material tends to approach a state of elastic behavior after many cycles of repeated loading. It would also provide a basis for determining the amount of slip involved in cyclic loading of an unbalanced type, such as being considered here.

Lacking such a theory, it does not appear possible to develop a rational method by which the effects of unbalanced loading (mean stress not zero) can be accurately predicted or quantitatively compared with those of balanced (completely reversed) loading. A few qualitative conclusions can be reached, however, by applying the foregoing ideas to special cases.

If the bulk behavior of the material were virtually elastic, but there happened to be a few "weak" crystals such as shown in Fig. 17, the amount of slip experienced by a crystal would be directly proportional to

the bulk stress (P/A or Mc/I). For unbalanced loading this would mean that the rate of growth of a crack would tend to be proportional to the **stress difference**, or range. This theory has often been proposed as a basis for the analysis of fatigue under unbalanced loading, but tests show that as the mean stress is increased, the allowable range of stress decreases. This would indicate that the degree of slip becomes greater as the mean stress is increased, while the range is held constant. It is also probable that a higher mean tensile stress would increase the number of atoms which become unbonded during each cycle.

Some actual test data for unbalanced loading will now be examined. Figs. 18 and 19 have been reproduced from Figs. 9 and 10 of Ref. 20, which reports the results of many tests made by the Battelle Memorial Institute for the N.A.C.A.

The curves for 75S-T6 aluminum alloy (Fig. 18) show a definite sag; this indicates an endurance limit of the type illustrated in Fig. 9 (b) for which the basic fatigue equation would have the form of Eq. 7 (a) or 7 (b). Fig. 19, for SAE 4130 steel indicates an endurance limit of the "cut-off" type (Fig. 9 c) with a basic fatigue equation in the form of Eq. 4.

For both materials it can be seen that over a large range of R the curves have about the same slope on the log-log plot. This shows that the amount of slip involved is a function of s^x, as previously assumed for balanced loading. At high values of R, however, the slope appears to decrease. This effect becomes noticeable in the region of the tensile yield stress and it indicates a change in the nature of the action involved. On the basis of the stress-strain analysis presented in Chapter 16 of Ref. 7 it would appear that **time effects** begin to become important when the

maximum stress level exceeds values in the neighborhood of the yield stress.
Since a lower slope represents a higher value of x (see Eq. 4 c) this would
indicate that the maximum stresses are getting well beyond the knee of the
stress-strain diagram. Beyond this point the stress-strain loop probably
widens very rapidly with increasing stress-level, becoming a quarter-loop
(one-time loading) at the value of the ultimate tensile stress for
the rate of load (or strain) application involved.

If any attempt were made to correlate fatigue behavior with ultimate
tensile stress in the high range of loading it would seem necessary to work
with true stress rather than the nominal value. This will be discussed later.

Since addition on a log-log chart actually represents multiplication,
it can be seen from Figs. 18 and 19 that the effect of a mean stress other
than zero (R = const.) is to multiply the value of N by a constant factor.
This factor is evidently a function of R. Eq. 4 a would therefore be
changed to

$$N = \frac{K_R B}{s^x} \tag{26}$$

where

$$K_R = f(R)$$

By referring to the derivation of Eq. 4 a it can be seen that $1/K_R$
must represent the ratio between the number of atoms unbonded in unbalanced
loading and the number unbonded in balanced loading, at the same maximum
stress level. The increase in fatigue life, as the minimum stress is raised,
must therefore be caused by a narrowing of the stress-strain loop. The
increase in mean stress will probably tend to offset this to some extent,
as previously noted.
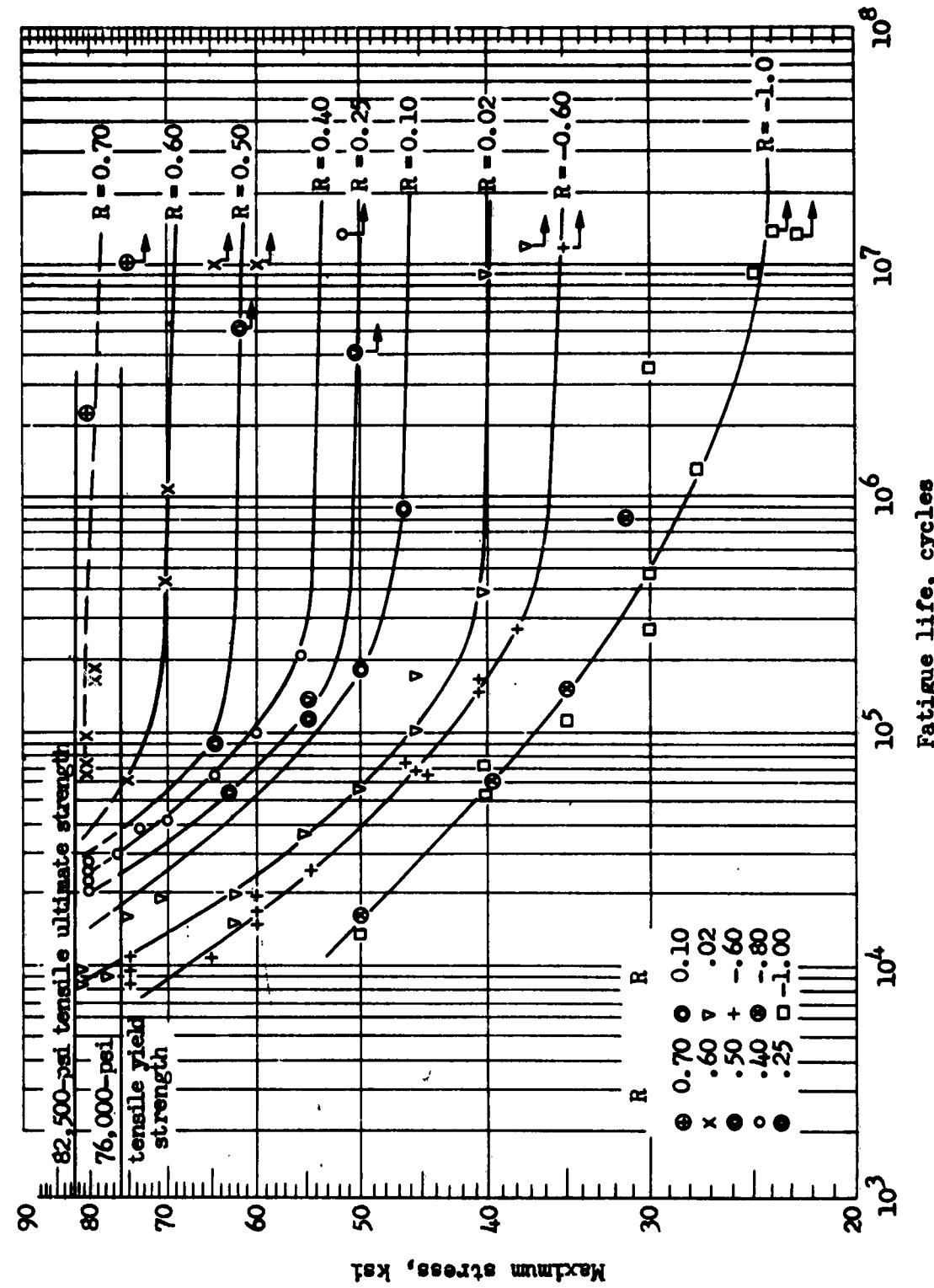
Fig. 18 — Results of Fatigue Tests at 1100 Cycles Per Minute on 75S–T6 Aluminum Alloy

(From NACA TN 2324)
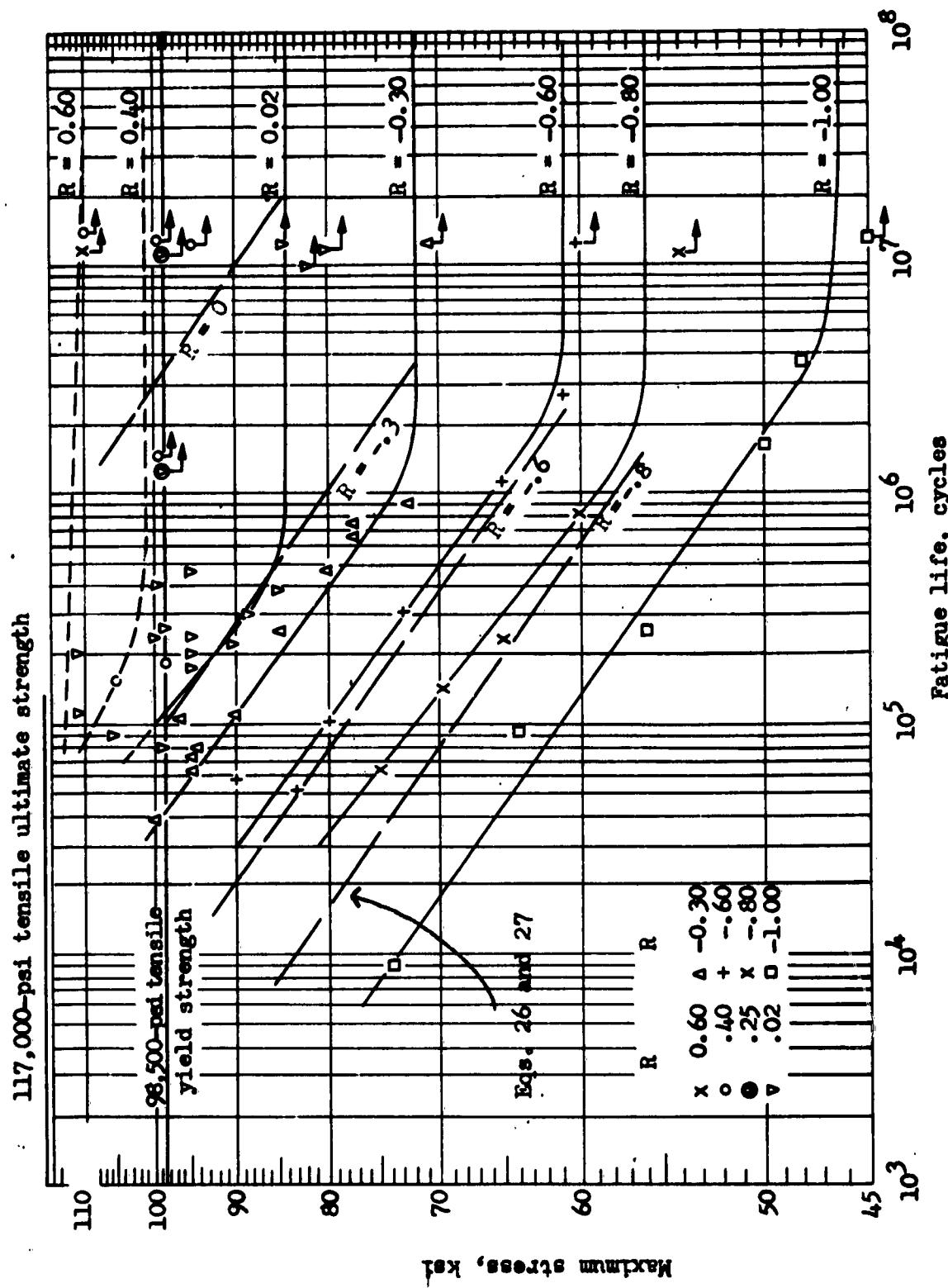
Fig. 19 - Results of Fatigue Tests at 1100 Cycles Per Minute on Normalised SAE 4130 Steel.

(From NACA TN-2324)

In view of the fact that test results for unbalanced loading are rapidly becoming available to the designer there is no urgent need for a theory which will predict accurately the fatigue curves for various values of R. However, it is of interest to see whether an approximate analysis, based on the foregoing principles will give results which have the proper trend. To this end, several schemes were tried in an effort to express the magnitude of the inelastic strain in terms of R. One of these is shown in Fig. 20. Only the inelastic strain is shown in these diagrams.
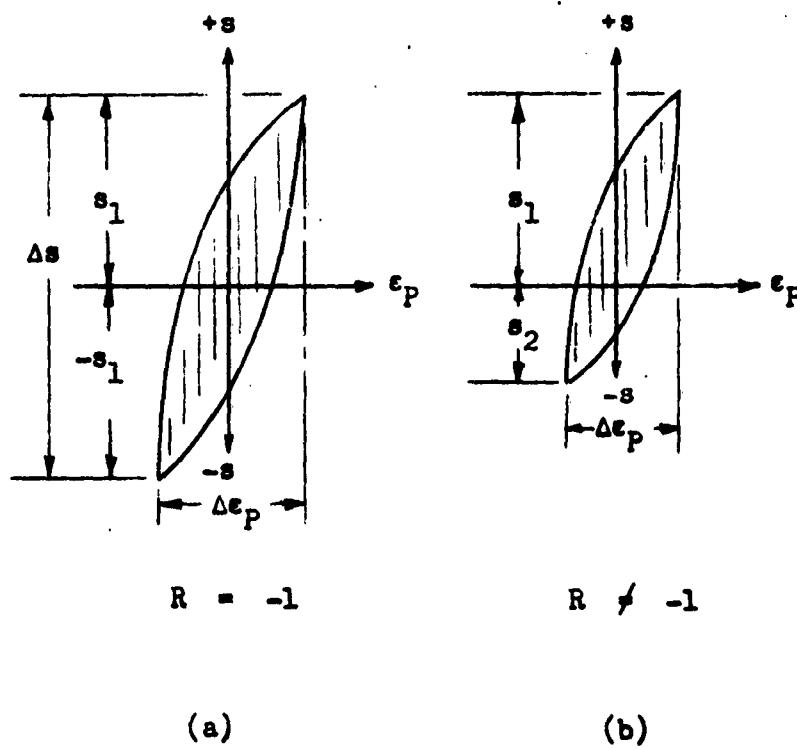
Fig. 20   Stress-Strain Loops (inelastic strain only)

For the case shown in (b) the ratio of plastic strain to that for balanced loading (a) is obtained as follows:

$$\frac{1}{K_R} = \frac{e_R}{e_B} = \frac{c\,(s_1 - s_2)^x}{c\left(2s_1\right)^x}$$

$$\frac{1}{K_R} = \frac{(1 - R)^x}{2^x}$$

$$K_R = \frac{2^x}{(1 - R)^x} \tag{27}$$

where $\quad R = \dfrac{s_2}{s_1}$

To use Eq. 27, the value of N for balanced loading at a given stress is determined and then multiplied by K_R to find the value for any given R.

$$N_R = K_R N_B \tag{28}$$

where
N_B = value for balanced loading.

The curves obtained by this method are indicated on Fig. 19 by dashed lines, for four values of R. For R = -.8 and -.6 the computed values of N_R are lower than the test values, but for N = -.3 and 0 they are higher. The overestimation of fatigue life increases rapidly as the mean stress increases. This would indicate that the effects of mean stress are not restricted to the effect on the stress-strain loop, but also tend to increase the rate of crack growth directly, as previously mentioned.

Various modifications and improvements of the above elementary analysis suggest themselves. For example, the type of fatigue equation represented by Eq. 7 would give different results. Different assumptions for the shape of the stress-strain loop could be made. A factor relating rate of crack growth to the value of mean (or minimum) stress could be included in the derivation of the fatigue equations (this appears to be the most logical step to take). No attempt will be made, in this paper, to modify the foregoing analysis of unbalanced loading so as to obtain better agreement with the test results. This is an interesting field for research, provided that the analysis is made on the basis of a logical theory of inelastic behavior, as previously suggested. On the other hand, it would add little to our knowledge of fatigue merely to include more adjustable factors in Eq. 27, in order to make it fit the test data.

As explained in Part 2 the endurance limit may be interpreted as the maximum value of repeated stress at which an elastic state may be maintained, or a "quasi-elastic" state developed before crack growth entered the relatively rapid stage. For unbalanced loading the rate of crack growth is reduced because of the reduction in the amount of inelastic strain, represented by the narrowing of the stress-strain loop. This alone would tend to permit a higher level of maximum stress to be applied. In addition, the narrowing of the initial stress-strain loop also means that fewer cycles would be required to attain the quasi-elastic state. Both of these effects would result in an increase in endurance limit with increasing mean stress.

One of the most convenient methods of plotting the results of fatigue tests under unbalanced loading is the Goodman* diagram, in which the value of maximum stress is plotted against the mean stress. By cross-plotting from the s-N diagram (Figs. 18 and 19) a curve may be obtained for any given value of N. (For examples see Fig. 8, p. 35 of Ref. 1, or Figs. 110 to 113, of Ref. 2.) Goodman's original hypothesis was that the curve representing the endurance limit (N = ∞, or some very large number) would vary linearly with mean stress. In order to include the fact that failure must occur when the mean stress reaches the ultimate tensile stress the curves are usually extended to include this point. Fig. 21 shows the values of maximum stress at 10^7 cycles, plotted in this manner from Figs. 18 and 19.

The assumption of a linear variation of maximum stress between that for completely reversed loading and that for one-time loading (ultimate tensile stress) is at least reasonable. In the two cases shown it would be on the

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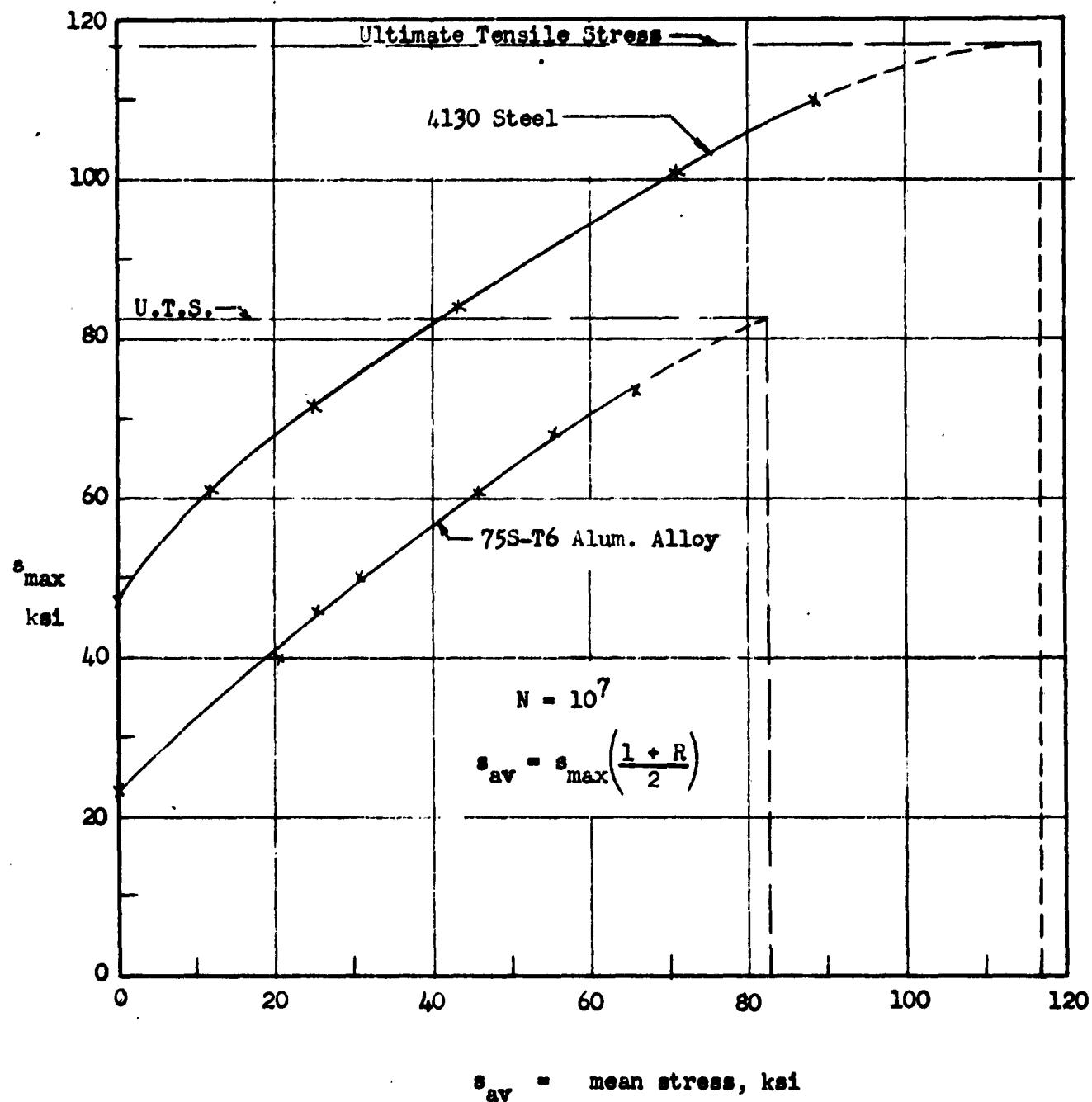*Usually called the modified Goodman diagram.

Fig. 21. Goodman Diagram for Endurance Limit, Obtained from Fig. 19.

safe side. The upper region of the curves would become more nearly straight
if the <u>true stress</u> were used instead of the nominal stress. (Since there
is no necking effect in fatigue it is illogical to compare fatigue strength
with ultimate tensile strength on the basis of nominal stress.) In using
<u>true stress</u> the upper ends of the curves would be extended along a straight
line through the origin, to the value of true stress corresponding to the
nominal ultimate tensile stress (<u>not</u> to the true stress at fracture).

The Goodman diagram (Fig. 21) may be physically interpreted as follows.
As the mean stress is raised, the action takes place at a smaller range
and at a higher level on the stress-strain diagram. The range must decrease
in order that the integrated rate of crack growth will remain the same,
thereby insuring the same value of N (or complete stoppage of crack growth
in the case of a true endurance limit). When the maximum stress reaches
the ultimate tensile stress (at maximum load) the amount of slip which
occurs on the first application of load is so great that the nominal stress-
strain diagram becomes horizontal, or even drops (indicating necking).
This means failure at "one-quarter cycle"[*]: no reversal of loading is then
possible. The point representing the ultimate tensile stress should there-
fore not appear on a diagram which indicates the allowable maximum stress
for a large number of reversals, or for complete absence of failure. It
may, however, be used as a hypothetical limit.

The same line of reasoning shows that the typical s-N curve should
actually begin at one-quarter cycle and at the (true) ultimate tensile
stress.

One point is clear from the above reasoning: <u>for ductile materials</u>
<u>both fatigue failure and ultimate failure can be placed on the same</u>
<u>physical basis</u>: both result from inelastic strain. It is also clear that

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[*]It appears that this refinement is of little significance; see Supplement.

the value of the true stress at actual failure, after necking has occurred, has virtually no significance in structures which are expected to fail in a ductile manner under one-time loading, or in fatigue under repeated loading.

Further investigations along these lines might reveal a method of analyzing repeated stresses of variable amplitude in such a way that one-time loading is automatically included. For example, the stress corresponding to one-time loading could be given a very high weighting factor in the expression for reduced stress (Part 3). The final result would be that the variable loading spectrum would cause some reduction in the allowable tensile stress used for the "one-time" loading analysis. The possibility of treating failure at the ultimate tensile stress as a special case of fatigue is being investigated and will be discussed in a subsequent paper on stress-analysis methods.

PART 5 - EFFECTS OF STRESS-CONCENTRATION, TEMPERATURE, AND RELATED FACTORS

It is well known that most fatigue failures which occur in service are the result of a combination of fatigue and stress-concentration. The insidious thing about this combination is that a stress-raiser (such as a notch or hole) may have little or no effect on the static strength (one-time loading) but may greatly reduce the allowable stress required to ensure the desired lifetime, in terms of cyclic loading.

In view of the large amount of work that has been done in this field, the present discussion will be directed toward interpreting the effects of notches in the light of the ideas and theories which have been presented up to this point. Only a few of the more recent papers on notch effects will be referred to.

The first step that must be taken toward clarifying the effects of stress-raisers is to direct attention away from stress and toward strain, particularly inelastic strain. It has been shown that a logical theory of fatigue can be developed from the principle that a crack forms and grows through the gradual unbonding of atoms as the result of reversals of inelastic strain. The adverse effects of a stress-raiser indicate that the rate of growth of the crack is increased. This could be due to two causes, as mentioned in Part 4: increased slip or the increased tendency to unbond caused by higher tensile forces. From available indications it appears that both effects occur, but that the major effect is the increase in slip.

In order to visualize stress concentration effects it is convenient to use a simplified presentation which the author developed in teaching

strength of materials. Figure 22 shows a strip of constant thickness, under the action of a tensile force P. At the ends of the strip, or at some considerable distance away from the notch, it may be assumed that the distribution of force is constant over the width, B. If this width is now
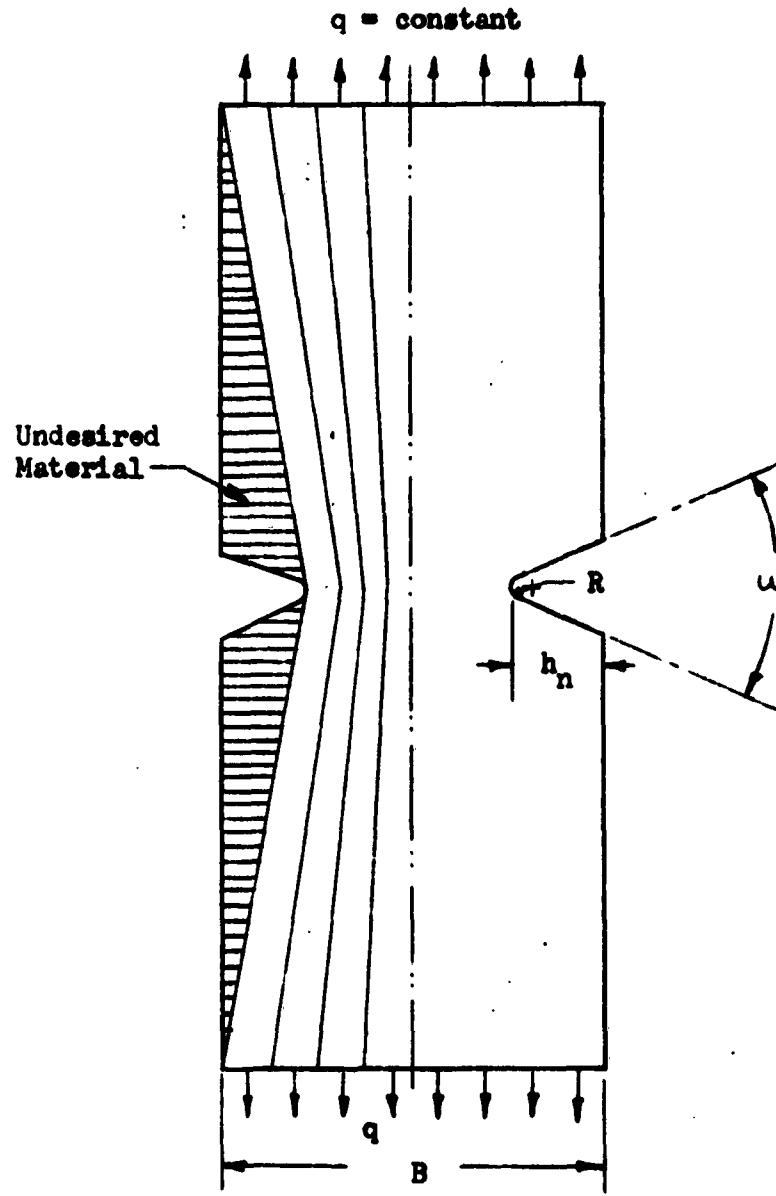
Fig. 22 Effect of a Notch

divided into a number of elements, ΔB, the force in each element will be

$$q = \frac{\Delta P}{\Delta B} \tag{29}$$

(The stress will be equal to q/t, where t is the thickness.)

Now if it is desired to eliminate the stress-concentration effect of the notches, the object will be to achieve a uniform distribution of force over the width at the root of the notches. Hence this width is divided into the same number of elements and "streamlines" are drawn to join these elements with those at the end of the strip. If the force q remains the same in each streamline, the desired uniformity of stress will have been achieved.

It is immediately apparent that there is a large wedge of "undesired material" adjacent to each notch. If these wedges are cut away, and if the notch is at a considerable distance from the ends, there will be no appreciable notch effect.

Now assume that after this material has been cut away the specimen is elongated by application of the tensile force. Next imagine the system of forces that must be applied to a wedge in order to stretch it so that it will fit the elongated specimen in its original relationship. This will require a transfer of force across the "cut" section; it is important to note that this represents a shearing action. The amount of force needed to stretch the specimen will be greatest at the thickest end of the wedge, adjacent to the notch. The transfer of force will reduce the stresses in the central portion between the notches. At the root of the notch a large flow of force will be added to the streamline, since the two thick ends of the wedges will tend to equalize their tensile forces through the material adjacent to the

notch. By definition, the streamlines must each carry a constant value of q; therefore the ones near the root of the notch must be narrowed and those near the center of the strip will become wider, approaching the width of the streamlines at the ends of the specimen as the plate is made wider relative to the notch depth.

The exact location of the streamlines, in the elastic range, can be obtained by an elastic analysis, or by photoelasticity. The main point to be observed in the above analysis, however, is that relatively high shearing stresses must exist along the "cut" lines, in an actual (uncut) specimen. If these shearing stresses were to cause slip at a relatively low level of stress, the effect would be to permit the wedge to slide along the "cut" line, to some extent. In fact a complete cut may be thought of as representing slip at zero shear stress. (Various ways of relieving notch effects, such as making saw-cuts or drilling holes in the "undesired material", obviously permit the wedge to elongate more easily and therefore reduce the transfer of force into this region.)

At the root of a sharp notch the abnormally high tensile stress will cause shear slippage over a small area. This will not have any appreciable effect on the forces in the wedge. If the material near the root continues to slip without any substantial increase in stress, the forces required to stretch the wedge must be distributed farther inward. It can be seen that the effect of shear slippage at the root of a sharp notch is to level off the peak of the stress-distribution curve. This is of course well known and is illustrated, for example, on page 64 of Ref. 21. The amount of slip at the root of the notch is not reduced by this process. In fact it must become

greater as the applied tensile force increases. Since it has been shown that fatigue is probably the result of slip, it would appear that the primary adverse effect of stress-raisers is the result of increasing the degree of slip. We should therefore be concerned with strain-raisers, not stress-raisers.

If the elastic stress-concentration factor, K_E, is used to determine the stress at the root of the notch and this increased stress is then used in connection with the basic $s-N$ diagram, the results tend to underestimate the fatigue life (this is on the safe side). A "fatigue-strength reduction factor", originally proposed by Hartmann (Ref. 22), can be determined by working backward from fatigue tests of notched specimens. This factor, K_f, is found to be less than K_E, particularly for high stresses and sharp notches (Ref. 21). It is generally agreed, therefore, that there is an alleviating effect caused by the inelastic behavior at the root of the notch.

From the simplified analysis based on Fig. 22 it would appear that the alleviation effects of slip would be more likely to come from the region near the "cut" line, in an area slightly removed from the notch, where the shear stresses will be high. This idea is supported by the findings of Sir Richard Southwell, who has shown by relaxation methods that (for a flat-topped stress-strain diagram) a plastic region will form first at the root of a notch, but that as the loading is increased another separate region will begin to form at some distance away from the notch. These regions merge at high stress levels. (See p. 247 of Ref. 23.)

Turning attention now to the shape of the notch itself, the most important dimension in Fig. 22 would appear to be the notch depth. This

controls the amount of material in the wedge. But the notch angle, ω, also affects the amount of material in the wedge; as it widens, the notch approaches the "cut" lines, for which the stress concentration effects would be negligible.

For the Vee-type notch shown, it would seem that the value of the radius at the root would have little effect on the amount of slip that occurs in the region at the root of the notch. It is true that a small radius will cause a high elastic stress-concentration factor. However this effect is highly localized and would be quickly "washed-out". It would simply mean that localized slip would start at a lower stress.

It is important to realize that the slip on which the rate of crack growth depends is probably the result of a stress-distribution which extends inward some distance from the notch. Because of continuity of the material, a large highly-localized slip cannot exist adjacent to a region of low slip. This may be the physical explanation of the "Neuber constant" (Ref. 24): this constant could represent the effective width over which the slip acts.

A concise description of the Neuber constant is quoted below, from Ref. 25:

"In the classical theory of elasticity, the material is considered as a continuum. Pointing to the fact that engineering metals have a granular structure, Neuber stated that this concept must be abandoned when a stress gradient is present. He proposed instead the concept that the material is an aggregate of "building blocks" and postulated that no stress gradient can develop across such a block; the quantity A is the half-length of a block. Neuber stated that the length A should be

considered as a new material constant and that it must be determined by experiment."

Reference 25 shows that the Neuber constant can be correlated with the ultimate tensile strength so as to give improved results. (For the steels investigated this value varied from about .017 inch at an ultimate tensile stress of 50,000 psi to about .0004 inch at 190,000 psi.) Since, in a certain material, higher ultimate tensile stresses are obtained primarily by reducing the amount of inelastic strain, it seems logical that the width of the region in which most of the slip occurs would decrease with increasing ultimate tensile stress.

In the extreme case of a crack caused by fatigue, the "notch radius" would appear to approach zero. Actually the "crack" would be represented by a path of unbonded atoms and it is quite possible that there is a more or less gradual transition from zero bond to complete bond at the end of the crack. In any case, the primary action of this type of stress-raiser would be much the same as that of a notch with a small radius; the depth of the crack would be the most important factor.

In Part 4 it was shown that in cyclic loading a "weak" crystal would be compelled to slip and that the degree of slip would be governed by the overall strain in the surrounding medium. This was a direct application of the continuity principle that plane cross-sections remain plane. The same reasoning can be applied to a notched specimen, but now it is evident that cross-sections, originally plane, will be considerably warped in the region of the notch or crack. Under the action of a tensile force the free edges of the notch will tend to move apart much farther than they would if

they were bonded together in an unnotched specimen. This permits the strain at the base of the notch to exceed the average strain by considerable amounts, thus creating a "strain-raiser".

It is interesting to speculate on what would happen if the direct tensile stress at the end of a crack became equal to the ultimate cohesive bond between two atoms. The attractive force between the atoms would drop off with increasing distance between atoms (see Fig. 9.9 of Ref. 3). Now if the crack is assumed to widen further the decrease in tension between the first pair of atoms must be carried by the next pair. This would cause them to exceed their ultimate bond stress and this action would go on very rapidly until the crack had spread through the material. This unstable condition is evidently the picture of <u>brittle</u> <u>fracture</u>. If the material were polycrystalline, it might be argued that the above reasoning would apply only to crystals having a certain orientation and that the crack would be stopped by encountering "weaker" crystals which would undergo slip. But if the "strong" crystals were surrounded by weaker ones, as in a random distribution of crystals, the tension stress required to break the bond by direct separation could not be developed. Hence the failure would have to be of the ductile type (1/4 cycle) or of the fatigue type (many cycles). Since the inelastic strain in the "weaker" crystals depends on the temperature and the rate of loading, it is easy to see why these factors have such important effects in determining whether a failure is of the ductile or brittle type. (See Ref. 1 or 3 for further discussion of this.)

The following principle can be stated, for possible further investigation and verification. <u>The</u> <u>essential</u> <u>difference</u> <u>between</u> <u>fatigue</u> <u>failure</u> <u>and</u> <u>brittle</u> <u>failure</u> <u>is</u> <u>that</u> <u>in</u> <u>fatigue</u> <u>the</u> <u>atomic</u> <u>bonds</u> <u>are</u> <u>broken</u> <u>by</u> <u>a</u>

shearing action while in brittle failure they are broken by direct tension.

Another factor that has an important effect on the conditions at the root of a notch is the lateral or radial stress which must exist for equilibrium wherever axial (tension or compression) stresses are transmitted around a curve. For cylindrical specimens having a notch which encircles the specimen the three-dimensional stress situation for an element a small distance inward from the notch is such as to reduce the maximum shearing stress. The degree of this reduction will depend on the degree of curvature: for very small radii the shear stress will be appreciably reduced. This well-known effect would tend to reduce the degree of slip in a region close to the root of the notch. Reduction in the notch radius therefore appears to have two opposing effects: (1) the elastic stress concentration factor is increased, (2) the maximum shearing stress is reduced. Such effects would have to be considered in a detailed analysis of the problem.

The effects of residual stresses, already discussed in Part 4, must also be considered. (See Fig. 26, Ref. 21.) In general it would appear that a "stress-raiser" will magnify these effects in proportion to the degree by which the inelastic strain is magnified. Actually, however, the residual stresses which result from reversal of loading are merely evidence that the material which underwent slip is being forced to slip again in the opposite direction. Hence it would seem that they do not enter the analysis directly.

Residual stresses of considerable magnitude, produced by a one-way loading before the cyclic stresses are applied, can have relatively large effects under certain conditions, as shown by Rosenthal and Sines (Ref. 26).

For high-strength (high yield-stress) materials it is possible, in effect, to change the mean stress level considerably by prestressing. If this is done in the proper direction, and if there is no possibility of a large subsequent loading in the opposite direction, this technique might be used to reduce the probability of a fatigue failure. This situation exists in aircraft wing structures (outboard of landing gear attaching points) and in many stationary structures. On the other hand, a residual tensile stress would tend to raise the mean stress level and would, therefore, have an adverse effect on fatigue life, especially if the cyclic stresses were not great enough to "wash out" the unfavorable residual stress.

From the design point of view, information on the effects of various types of stress-raisers is rapidly being obtained in the form of $s-N$ diagrams for different levels of mean stress (see Refs. 21, 27, and 28). A portion of Fig. 6 from Ref. 21 is reproduced here as Fig. 23.

Fig. 23 Results of Axial-Load Fatigue
Tests on Notched 24S-T3 Alum-
inum Sheet Specimens - $K_t = 4.0$
(From NACA TN 2389, Fig. 6)
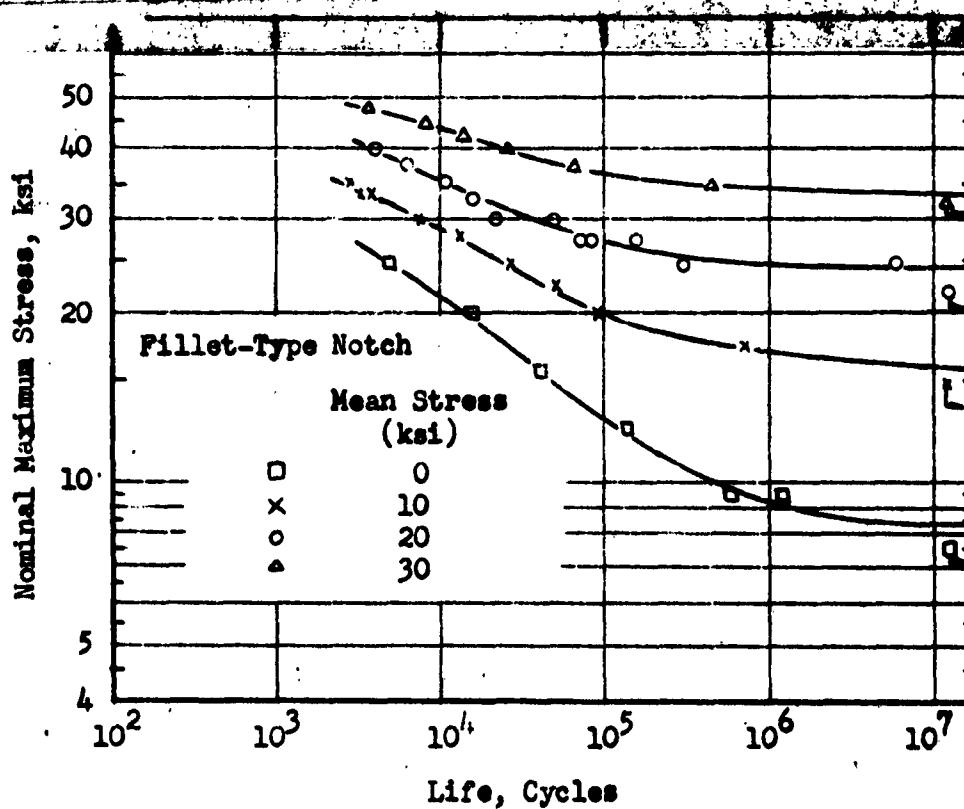
The type of stress-raiser used for these tests is shown in Fig. 24.
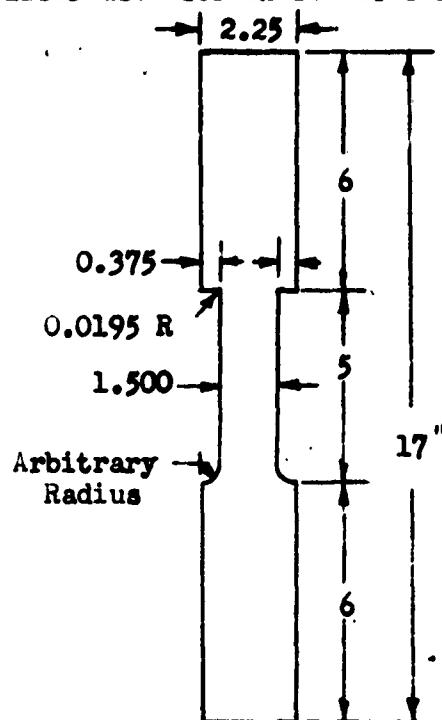
Fig. 24 Notched Fatigue Test Specimen
with $K_t = 4.0$
(From NACA TN 2389, Fig. 2)

Figure 23 shows that the effect of increasing mean stress, for notched specimens, is not only to raise the fatigue curve[*] but also to decrease the slope. This same effect was evident in nearly all of the tests of this type. This indicates that the higher stress level now changes the exponent x in the expression used to relate inelastic strain to stress (Eq. 2). In Part 4 it was shown that for unnotched specimens a change in mean stress level did not appreciably affect the slope until the maximum stress approached the yield stress. This difference in behavior is no doubt due to the effect of the notch in raising the mean stress level, or more accurately, the strain level, at which the action takes place. Severe notch effects tend to correspond to the higher (and flatter) curves of Figs. 18 and 19.

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[*]Note that the curve shows the maximum stress; the variable stress decreases with an increase in mean stress.

The curves of Refs. 21, 27, and 28 also exhibit, in general, a tendency to converge in the vicinity of one-quarter cycle and at a stress considerably above the nominal ultimate tensile stress for the material tests. Since the true stress corresponding to the ultimate tensile stress is also likely to be considerably higher, this would seem to support the idea mentioned in Part 4, that the true stress at ultimate load (not at failure) might represent the limiting case of fatigue at one-quarter cycle.

The effects of temperature can be introduced into fatigue analysis in a qualitative manner by considering two things: (1) the effects on the inelastic strain (stress-strain loop), (2) the possible helpful effects of high temperatures on re-bonding after slip. In the first case the obvious answer would be to consider the material as a new material at each different temperature level and to work with the stress-strain diagram which results. As the temperature is increased, however, a single stress-strain diagram becomes inadequate to describe the inelastic strain behavior. (See Ref. 7, Ch. 16.) Time effects now become important, especially in the higher stress ranges. It may therefore be expected that the rate of loading (or straining, if controlled strain is used) will have a much greater effect than it does at lower temperatures. Lower rates of loading should produce more slip per cycle. In the extreme case of a very low rate of loading the specimen might fail on the first load application, by "stress rupture" (see Ref. 7, p. 276). This would replace the ultimate tensile stress used for time-independent materials.

For loading at a mean stress greater than zero it would appear likely that the stress-strain loop would "move to the right" along the stress-strain diagram and that failure would be very similar in nature to that for one-time loading.

There is a possibility that at very high temperatures some materials would tend to rebond under the action of the compressive forces exerted during stress reversal. If this happened, it would be impossible for a crack to form by successive unbonding of atoms. Metallurgical effects would also have to be considered, at elevated temperatures.

These ideas are suggested in order to show how the proposed theory of fatigue could be used to predict the effects of elevated temperatures. A careful survey of existing data may indicate whether the predictions are substantiated by tests. If existing test data are inadequate, it would be desirable to conduct some special tests at elevated temperatures, using different rates of loading.

The idea of rebonding at higher temperatures leads to an examination of the helpful or detrimental effects of the surrounding medium. Corrosion fatigue can be explained by the proposed theory, by noting that the atoms exposed on each reversal of slip would be chemically attacked, thus decreasing the probability of rebonding. The basic fatigue equations could be modified to provide for this. It would seem likely that the factor C in Eqs. (2) and (3) would be increased, thus reducing the value of the constant B in Eq. (4) and causing the $s - N$ diagram to be lowered by a constant amount on the log-log plot. It is possible also that the exponent x would be affected because of the adverse combination of corrosion and higher stress levels. This would change the slope of the $s - N$ diagram on the log-log chart. Both effects have been observed in tests. Very large reductions in the endurance limit are known to occur when corrosion effects and notch effects are combined (see p. 70, Ref. 2). Softer alloys appear to give

better results than hard ones under these conditions, even though their
endurance limit under normal conditions may be relatively low. This could
be explained by the fact that in a notched specimen the corrosive action
speeds up the unbonding process, but does not affect the slip which occurs
some distance away from the notch.

The possible gain in fatigue strength under conditions conducive to
rebonding would seem to be an interesting subject for experimental investi-
gation. Tests could be made in non-corrosive media and at various tempera-
tures. Non-oxidizing materials, such as gold, could be investigated.

The significance of the transition temperature in fatigue has been
clearly discussed (Paper 11 of Ref. 1) by C. W. MacGregor, one of the
leading investigators of this phenomenon. The transition temperature, as
obtained in the slow-bend tests at M.I.T., is defined as "the highest tempera-
ture under which no microscopic plastic flow occurs under fixed conditions
of constraint and strain rate". At temperatures below the transition
temperature a "brittle" failure is obtained; above this temperature a
"ductile" failure is obtained. In the above definition "constraint" refers
to notch effects. It has been found that the transition temperature is
increased by cyclic loading in a fatigue test.

Many of the observations and suggestions summarized by MacGregor in
Ref. 1 appear to fit perfectly into the picture of fatigue as presented up
to this point. For example, it was assumed that cracks start from the
beginning of loading, due to inelastic slip, and that the growth of the
crack depended on the width of the stress-strain loop. Under some conditions
this loop may become a straight line before the crack becomes observable,

resulting in an endurance limit. This represents elastic behavior. The narrowing of the stress-strain loop should certainly affect the transition temperature, because the more nearly the elastic condition is approached the higher the temperature must be to produce sufficient localized slip to prevent brittle fracture. This would explain how the transition temperature can be raised by cyclic loading below the endurance limit.

It is also generally agreed that fatigue cracks will raise the transition temperature; however, some investigations showed such effects when no cracks could be detected. It would appear that the methods used to detect the cracks could not have revealed the presence of a crack produced by unbonding in its early stages, as previously noted.

A significant statement appears on page 247, to the effect that at a stress 15 percent below the endurance limit a crack continued to spread and that the rate of crack growth decreased as the number of cycles increased. This fits in with the idea (stated by Gensamer, p. 3, Ref. 1) that "fatigue is a race between hardening and damage". Below the endurance limit the "hardening" (represented by the narrowing of the stress-strain loop) wins the race over the "damage" (growth of the crack).

PART 6. GENERAL DISCUSSION

The essential difference between the proposed theory of fatigue and most of those which have preceeded it lies in the mechanism of crack formation, in which the crack is formed by progressive unbonding of atoms as the result of alternating slip. This relates fatigue to the amplitude of inelastic strain which occurs in repeated loading, rather than to some value of stress at which a crack would begin to form. It also leads to the idea that the crack may start at the very first load reversal, even though the crack may remain invisible until it has progressed to a certain depth, or until it has produced fragmentation.

The idea that fatigue cracks are connected in some way with slip is not new. Timoshenko (Ref. 4, p. 461) refers to a paper by Ewing and Humfrey, dated 1903, and states: "On the basis of such investigations the theory was advanced that cycles of stress, which are above the safe range, produce slip bands in individual crystals; if we continue to apply such cycles of stress there is a continual sliding along the surfaces accompanied by friction, similar to that between sliding surfaces of rigid bodies. As a result of this friction, according to the theory, the material gradually wears along the surfaces of sliding and a crack results". Since slip bands were also found to occur below the endurance limit of the material, it was concluded that the appearance of slip bands cannot explain the mechanism of fatigue cracks. It is also evident that the theory proposed by Ewing and Humfrey would require that torsion fatigue cracks occur in the plane of maximum shear stress, but this is not generally observed in tests.

The theory proposed in this paper does not depend on the wear or friction developed, but rather on the unbonding of atoms at a free surface, as described in Part 1. The presence of slip at stresses below the endurance limit is explained by the idea that unbonding does occur below this limit, but that the gradual narrowing of the stress-strain loop eventually stops crack growth completely.

Experimental evidence to support this theory may be found scattered through the voluminuous literature on fatigue. Only a few references will be cited here. On page 266 of Ref. 1, in a paper by Teed, there appears a photomicrograph (from the Royal Aircraft Establishment) showing a fatigue crack on a slip plane. Major Teed states, in part, "In an alloy below its recrystallization temperature, it has long been established that a crack engendered by cyclic stresses ------- arises and spreads along a slip plane, ---- The spread of the crack across the intercrystalline boundary is somewhat, though not greatly, slowed down by it. The crack then usually continues to develop along a slip plane in a suitably oriented adjacent crystal ------- The original crack generally starts in the plane of maximum resolved shear stress which happens to be approximately parallel to one of the axes of the crystal in which it occurs." Major Teed also makes a number of other interesting observations which appear to be in line with the present theory. On page 80 of Ref. 1, Peterson also shows photomicrographs of fatigue cracks (from Hull) which progress across the crystals in a zig-zag manner.

The possibility of change of direction of the crack from crystal to crystal explains one of the puzzling features originally encountered in making a working model of crack formation by cutting an array of "atoms" on a 45 degree line and sliding the two halves of the paper back and forth

(see Fig. 2). The crack, of course, remained on the 45 degree plane, while actual fatigue cracks appeared to form at 90 degrees to the direction of axial loading. It would appear that although the general direction of the crack is normal to the applied tensile stress, the actual crack progresses in the zig-zag manner described above.

Another apparent obstacle was encountered in applying the theory to torsion of a round bar. On the basis of slip alone, one would expect the crack to form along the plane of maximum shear stress, normal to or parallel to the axis. But under alternating slip in this plane no atoms could become unbonded by slipping out into the open air. There would also be no tendency to pull the atoms apart. Yet it is known that fatigue does occur in torsion. The answer is found by examining a torsion fatigue crack. The crack usually forms not along the plane of maximum shear stress, but normal to the plane of maximum tension stress, 45 degrees away. (See Fig. 5, p. 77, Ref. 1.) This tension stress causes a shearing stress (in another plane) which could force atoms to slip out along the surface and thus become unbonded. There would also be a tensile stress tending to promote unbonding.

The fact that torsion fatigue cracks do form in this manner is in agreement with the proposed theory and seems to strengthen the hypothesis that there must be a free surface. Occasionally, however, torsion fatigue cracks are observed to form in the plane of maximum shear stress. This could occur in accordance with the proposed theory only if there were an opening in the metal into which the material could slide. (Mr. George Sines has informed the author that there is some experimental evidence to this effect.)

The fatigue behavior of metals under combined loadings (such as bending and torsion) has been reported in various papers. If the reversed slip theory is applied to such cases, it would be expected that the fatigue strength would be a function of the maximum shear stress, provided that this

stress produced slip toward a free surface. For situations in which the shear stress effects must be averaged (integrated) in order to determine the overall stress-strain loop it would seem likely that the Batdorf-Budiansky theory (Ref. 19) would apply. The octahedral shear stress (shear-strain-energy theory) may also be regarded as a method of averaging and should be applicable in such cases. In Ref. 1, p. 1, Gensamer states that, for fatigue, "the effects of combined stresses, for ductile alloys, are adequately described by the shear-strain-energy hypothesis".

Mr. George Sines recently gave the author a short summary of a study he has made on fatigue under combined loadings. His conclusions seemed to be in general agreement with the above reasoning. He also brought out an important point which must be considered in analyses of this type. If one of the loadings (such as bending) remains constant while the other (such as torsion) is cyclic, the planes of maximum shear stress will not remain stationary. It would appear that the growth of the crack might be influenced by this. Methods which have been proved satisfactory for combined loadings which are in phase (such as the interaction curve method, Ref. 4, p. 435), may not be correct for loadings which are out of phase.

A point of special interest is the conclusion that, according to the proposed theory, fatigue could not occur in a truly elastic specimen. Again quoting Teed, from p. 259 of Ref. 1: "Brittle crystals such as quarts are in no degree subject to such a type of fatigue". This situation presents a dilemma to the metallurgist, in that fatigue could apparently be prevented by using "hard" materials, but the very thing that would eliminate fatigue (absence of slip) would also cause brittle fracture. This paradoxical situation is realised in ceramics, which are actually very strong (aluminum

oxide bundles have developed compressive stresses of over 300,000 psi), but which appear to have low tensile strength because there is no slip by which stress-concentration effects are alleviated.

One of the important premises of the proposed theory is that slip, having occurred first in one direction on a certain slip plane, will occur again in the reverse direction and on the same plane, upon reversal of the load. This may seem to conflict with conventional ideas of "strain-hardening", as sometimes used to explain fatigue and other phenomena. But the Bauschinger effect (lowering of the inelastic portion of the stress-strain diagram on reversal of loading; see Ref. 4, pp. 409 and 410) would seem to indicate that reverse slip does occur and that the shear stress required to reverse the slip if less than that required to produce it the first time.

A visible fatigue crack will appear to be concentrated on a particular slip plane, even though there is evidence of slip on many adjacent planes. This would indicate that minute "cracks" (unbonding) tend to form at first on many slip planes, but the ones which form most rapidly offer less resistance to slip and the action thus tends to concentrate on fewer and fewer slip planes as the cracks deepen. Craig (Ref. 9) states: "Failure often begins in more than one crystal at a time and at more than one place in a single crystal or grain". He also suggests that several microscopic cracks may join and thus produce failure at a more rapid rate.

The reversed slip theory seems to throw some light on the relationship between fatigue strength and ultimate tensile stress. As previously noted, a higher ultimate tensile stress (in a given base material) is usually obtained by reducing the degree of slip through alloying, heat treatment, cold-working, etc. Since it is assumed that the rate of crack growth depends

on the degree of slip, a reduction in slip should increase the fatigue life. This would explain the tendency for the fatigue strength to be roughly proportional to the ultimate tensile strength, when comparisons are made between different base materials or alloys.

But it was shown in Part 5 that the slip which controls crack growth at the root of a notch is governed by the overall slip in the specimen and is multiplied by a strain-concentration factor which depends on the shape and size of the notch. The alleviation of the notch-effect by inelastic behavior of the material must result from slip which occurs over a larger area, not that which occurs just at the base of the notch. The alleviating slip will also tend to be lower in magnitude than the crack-forming slip. Hence it is quite possible that an improvement in static strength, represented by a raising of the knee of the stress-strain diagram, might reduce the alleviating slip more than it reduces the crack-forming slip.

It would appear from this line of reasoning that the ratio between yield stress and ultimate tensile stress would have an important bearing on the fatigue strength. In a material with a low ratio (say two-thirds) the alleviating slip would be higher (in proportion to the crack-forming slip) than that for a material having a high ratio of yield stress to ultimate stress. This effect is known to exist, especially in the precipitation-hardening alloys (such as 75ST aluminum). Thus, it may be possible to predict, from the shape of the entire stress-strain diagram, how the fatigue strength will be affected by a certain metallurgical treatment or mechanical process. (See also the discussion in Part 4 in which it is suggested that the true stress at ultimate load might be a logical basis for comparisons involving both fatigue and static strength.)

The same line of reasoning might also be applied, in a more detailed manner, to the effects of alloying and heat-treatment on the behavior of individual crystals, in comparison with the overall effects on the aggregate. For example, a process which would increase the resistance of the individual crystal to slip, but which would provide considerable overall slip (ductility) might have a favorable effect on fatigue. The effects of crystal size and grain boundaries could also be analyzed in this manner.

Surface treatments, such as cold-rolling or shot-peening, have long been employed to improve fatigue behavior. They obviously help matters by producing a residual compressive stress in surface layers. When applied to localized areas, such as the root of a notch, they tend to create the desirable situation described above, in which the slip at the root is reduced without appreciably reducing the overall alleviating effects of slip in the surrounding material.

In Ref. 6 Foster stated, "A point that requires explanation is the total absence of fatigue failures in the soft annealed copper wire when subjected to varying strains in the order of twice the magnitude that would be necessary to fail a standard-size fatigue specimen of annealed copper in the number of cycles imposed". This phenomenon appears to be connected with the fact that the ratio of surface area to cross-sectional area increases rapidly as the diameter of a wire approaches zero. In Ref. 7 (p. 295) the author suggested that the exceptional tensile strength of very fine wires might be explained by the fact that the number of atoms that must be unbonded during slip would be greater than normal, relative to the number which may slip without unbonding. This same effect would probably apply to fatigue. Thus in the limiting hypothetical case of a single row of atoms the action would of necessity be purely elastic (since there would be no slip planes) and there could be no fatigue. It is also possible that the very high cohesive forces

which must be overcome in unbonding would permit a certain amount of alternating inelastic strain to occur in the bulk of the material, without causing any crack formation. Such effects would probably become more important as the diameter of the specimen is decreased, or as the ratio of surface area to cross-sectional area is increased in some other manner. (This suggests testing specimens having an abnormally high surface area.)

There appear to be possibilities of improving fatigue resistance by the creation of surface films · or coatings of high tensile strength. For example, in Ref. 29 Slifkin and Kausman report that surface oxidation of wires made from single zinc crystals produced a large increase in tensile strength. A recently developed method of applying a ceramic coating to "hot parts" for jet engines is reported by Solar Aircraft Co. to increase fatigue life (Ceramic Age, Aug. 1952, p. 46).[*]

A comparison of the proposed theory with the Orowan theory of fatigue (Ref. 8) will reveal certain points of similarity and disagreement. The major disagreement concerns the mechanism of failure. Orowan's theory is based on the idea that cyclic loading will cause localized strain-hardening, in the sense that the local stress in some small plastic region will continue to rise as a result of alternating strain. If this stress reaches a certain value, fracture is assumed to occur, thus starting a crack. The existence of an endurance limit is explained by the fact that as the stress rises in the plastic region the alternating strain decreases, finally becoming zero before the failure stress is reached.

Both theories are in agreement so far as the general explanation of the endurance limit is concerned, but on the mechanism of crack formation they are entirely different. (See Supplement for further discussion.)

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[*]This could have resulted from protection of the surface from oxidation.

The Orowan fatigue equation is developed first for a constant coefficient of strain-hardening (slope of stress-strain diagram) which results in a slope of unity on the log-log plot of the s-N diagram. In order to bring this into agreement with observed data, the coefficient is modified by arbitrarily making it proportional to the strain raised to some power. This may be compared to the use of Eq. (2) in the present theory and it has the same mathematical result: the fatigue curve now may have a slope other than zero on the log-log plot. The method by which the equation provides for an endurance limit is similar to that developed in Part 2, based on Fig. 9b.

It is worth noting that any theory of fatigue can be made to fit the observed data by including a term relating the fatigue life, N, to some power of the stress, such as s^X. Furthermore, almost any type of function may be used to describe the rate of crack growth. For example, the simplest possible equation for crack growth would be a linear one. Eq. (3) (Part 2) would then be replaced by

$$h_o = A \, s^X \, N \tag{30}$$

This would give the fatigue equation previously developed (Eq. (4a):

$$N = \frac{B}{s^X}$$

The use of a linear equation of crack growth would also result in the same methods for treating stresses of varying amplitude (for example, Miner's method, or Eq. (16) of this paper, in Part 3).

The use of non-affine functions for crack growth, such as Eq. (5),
appears to be necessary in order to obtain a satisfactory method of
treating stresses of varying amplitude. It is interesting to note that
Eq. (5) may be thought of as the derivative of Eq. (3), which was used
as the basis for the s-N equation. Eq. (3) therefore is related to Eq. (5)
in the manner of a potential.

The closest mathematical approach to the theory presented in this
paper appears to have been made by Vivian in Metallurgia, Jan. 1952,
(Ref. 30). He states that he investigated the "progressive propagation
of hair-cracks" and that "it seemed possible that N could be equated to
a power of exponential e containing a power of S which accounted for the
manner in which a rather flat-lying part of the S-N curve can be found
at which S can be called a 'safe stress' in many instances". Later he
stated "it seems to be an attractive possibility that, like the hardness
curve, the S-N curve is a certain type of modification of the true stress-
strain curve, the near-elastic and elastic portions of which (modified by
some unknown factor) are, in reality, an interesting part of the S-N curve".

In view of the fact that Vivian apparently did not have available any
definite mechanism of fatigue that would serve as a connecting link between
the S-N diagram and the stress-strain diagram, his intuitive understanding
of the problem appears rather remarkable and is in general agreement with
the theory proposed here.*

There are no doubt many other papers, at present unknown to the author,
in which some of the ideas assembled here have been presented. The refer-
ences cited will at least show that for a long time we have been very close
to obtaining a clear understanding of fatigue and that practically all the

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*Dr. Vivian has since informed the author that his suggestions were based on
an earlier paper, as yet unpublished.

necessary elements for a logical theory have been gradually discovered through hundreds of thousands of careful tests and analyses. The missing link in this vast array of information appears to be the mechanism of un-bonding through reversed slip. With the aid of this concept, failure under repeated loading can now be placed on the same level of understanding as failure under a single loading.

The equations employed or derived in this paper were used primarily to illustrate the above point. Many refinements and extensions are no doubt possible and will follow in due course, if the basic theory of the mechanism of fatigue is accepted. It is apparent, however, that the most immediate problem in fatigue is that of applying, in design and stress analysis, what we already know. It can also be shown that extreme refinement in theories or testing techniques can have very little effect on actual de-sign, in terms of weight saved or danger eliminated. (These matters will be discussed further in a subsequent report.)

PART 7 - RECOMMENDATIONS FOR RESEARCH

The theory of fatigue presented in this paper suggests some new approaches in research and development work; it also indicates where present research should be amplified or reduced. The following items will serve as a brief summary of the suggestions made in the preceding parts of the paper.

1. More emphasis should be placed on exploring the phenomenon of crack growth by means of the electron microscope and similar methods. The hypothesis of unbonding under reversed slip should be tested experimentally, so far as possible. (See p. 6.)

2. The influence of the shape of the (static) stress-strain diagram should be investigated, with a view to establishing a relationship between the Ramberg-Osgood exponent for plastic strain and the exponent for fatigue. (See p. 16.)

3. An attempt should be made to determine a relationship between the ultimate tensile stress (or modulus of rupture) and the intercept of the $s - N$ diagram at $N = 1/4$. The true stress at maximum tensile load (or bending moment) is suggested as an improved basis for comparison. (See p. 56.)

4. If the fatigue behavior can be correlated with the (static) stress-strain diagram (in items 3 and 4), this relationship should be tested by applying it to various metals having widely different characteristics. (Although a large amount of fatigue data are available, it is difficult to find reliable data for the true stress-strain diagram of the specimens tested.)

5. Emphasis in research should be shifted away from analysis of actual
    failure (in the tension test) and toward the analysis of factors
    which determine the maximum load (nominal ultimate tensile stress).
    (See p. 56.)

6. The effects on crack growth of crystal size and grain boundaries
    should be more thoroughly investigated. (See p. 79.)

7. The effects of notches and stress-raisers should be analyzed in terms
    of strain. The effects of inelastic strain in areas other than at
    the base of the notch should be more thoroughly investigated.
    (Relaxation methods appear to be most useful in such analysis.)
    (See p. 62.)

8. The effects of mean stress should be analyzed in order to determine
    whether there is a direct effect on the rate of unbonding. (See p. 47.)

9. The proposed method of determining a reduced stress for loadings of
    variable amplitude should be further investigated, particularly for
    situations typical of those found in aircraft structures (mean stress
    not equal to zero; stress-concentration factor greater than unity;
    stress spectrum of the type determined from gust data). (See p. 62.)

10. The effects of variable amplitude on the endurance limit (or lower
    portion of s-N diagram) should be explored. (Until this is done it
    appears advisable to extrapolate the s-N curve without regard to
    any apparent endurance limit obtained in loading of constant amplitude.)
    (See p. 37.)

11. Fatigue tests at elevated temperature should be conducted and analyzed
    to determine the effects of rate of loading. (See p. 69.)

12. The possible helpful effects of elevated temperature on rebonding
    should be investigated. (See p. 70.)

13. Special tests should be conducted, with a view to determining the influence of various surface conditions and environments on the rate of crack growth (examples: corrosive media, non-corrosive or protective media, surface treatments such as shot-peening, tests of gold specimens, etc.). (See pp. 79, 80.)

14. Additional fatigue tests of nearly elastic (brittle) materials should be conducted, to verify the hypothesis that repeated loading does not appreciably reduce the allowable stress. (See p. 76.)

15. The helpful and unfavorable effects of residual stresses (especially with stress-raisers present) should be more thoroughly investigated. In particular, the helpful effects of static-testing completed structures to limit load (approx. 2/3 of ultimate load) should be established and the limitations of such procedures should be determined. (See p. 66.)

16. Metallurgical research (in fatigue) should be directed toward discovery of methods by which large-scale plasticity (for relief of strain-raisers) may be obtained without causing high localized slip (which is the direct cause of fatigue). (See p. 78.)

17. Combined stresses should be more thoroughly investigated experimentally and an attempt should be made to apply the Batdorf-Budiansky theory of plasticity in the analysis of fatigue. (See p. 75.)

18. The fatigue of very fine wires, or members having large relative surface area, should be investigated. (See p. 79.)

19. The use of failure-indicating-wires should be further explored under various conditions and practical methods of application should be developed. (See p. 3.)

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NOTE: Recommendations on design and stress analysis of aircraft structures will be covered in a separate RAND paper, now in preparation.

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